

# Nonlinear waves in strongly interacting relativistic fluids

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## Abstract

During the past decades the study of strongly interacting fluids experienced a tremendous progress. In the relativistic heavy ion accelerators, specially the RHIC and LHC colliders, it became possible to study not only fluids made of hadronic matter but also fluids of quarks and gluons. Part of the physics program of these machines is the observation of waves in this strongly interacting medium. From the theoretical point of view, these waves are often treated with linearized hydrodynamics. In this text we review the attempts to go beyond linearization. We show how to use the Reductive Perturbation Method to expand the equations of (ideal and viscous) relativistic hydrodynamics to obtain nonlinear wave equations. These nonlinear wave equations govern the evolution of energy density perturbations (in hot quark gluon plasma) or baryon density perturbations (in cold quark gluon plasma and nuclear matter). Different nonlinear wave equations, such as the breaking wave, Korteweg-de Vries and Burgers equations, are obtained from different equations of state (EOS). In nuclear matter, the Walecka EOS may lead to a KdV equation. We explore equations of state such as those extracted from the MIT Bag Model and from QCD in the mean field theory approach. Some of these equations are integrable and have analytical solitonic solutions. We derive these equations also in spherical and cylindrical coordinates. We extend the analysis to two and three dimensions to obtain the Kadomtsev-Petviashvili (KP) equation, which is the generalization of the KdV. The KP is also integrable and presents analytical solitonic solutions. In viscous relativistic hydrodynamics we have second order partial derivatives which physically represent dissipation terms. We present numerical solutions and their corresponding algorithms for the cases where the equations are not integrable.

## I. INTRODUCTION

The elementary particles and their interactions are well described by the Standard Model (SM), which is an extremely successful theory [1]. In this theory matter is composed by quarks and leptons and their interactions are due to the exchange of gauge bosons. The sector of the SM which describes the strong interactions at the fundamental level is called Quantum Chromodynamics, or QCD [2]. According to QCD there are six types, or flavors, of quarks: up, down, strange, charm, bottom and top, and they interact exchanging gluons. Quarks and gluons have a special charge called color, responsible for the strong interaction. They do not exist as individual particles but, due to the property of color confinement, quarks and gluons form clusters called hadrons, which can be grouped in baryons and mesons. The former are made of three quarks, as the proton, and the latter are made of a quark and an antiquark, as the pion, for example. The quarks carry a fraction of the elementary electric charge and a fraction of the baryon number. The electric and color charges and the baryon number are conserved quantities in QCD.

The hadrons also interact strongly and their interactions have been traditionally described by a quantum field theory called Quantum Hadrodynamics, or QHD, in its different versions [3, 4]. According to QHD, in nuclear matter neutrons and protons interact exchanging scalar and vector mesons. This potentially complicated theory becomes quite simple in the mean field approximation, in which the meson fields are treated as classical fields. The different versions of QHD in the mean field approximation are called relativistic mean field models (RMF)[3, 4]. More recently, nuclear matter has been studied with Effective Field Theories, which incorporate the fundamental symmetries of QCD in hadron physics.

In the phase diagram of QCD, we can observe that under extreme conditions of very large temperatures and/or very large densities, the normal hadronic matter undergoes a phase transition to a deconfined phase, a new state of matter called the quark gluon plasma (QGP) [5]. Together with deconfinement, a second phase transition takes place: the chiral phase transition, during which chiral symmetry is restored and the light quarks (up and down) become massless. The hot QGP is produced in relativistic heavy ion collisions in the Relativistic Heavy Ion Collider (RHIC) at the Brookhaven National Laboratory (BNL) [6, 7] and even more in the Large Hadron Collider (LHC) at CERN. The cold QGP may exist in the core of compact stars [8].

According to our present understanding of the RHIC measurements, the QGP behaves as an almost perfect fluid and its space-time evolution can be very well described by relativistic hydrodynamics [9–11]. The discovery of this new fluid motivated innumerable theoretical works addressing viscosity in relativistic hydrodynamics [12–14]. At the same time, more sophisticated measurements made possible to study the propagation of perturbations in the QGP. We may, for example, study the effect of a fast quark traversing the hot QGP medium. As it moves supersonically throughout the fluid, it may generate waves of energy density or baryon density [15]. It may be even possible that these waves may pile up and form Mach cones, which would affect the angular distribution of the produced particles, fluid fragments which are experimentally observed.

The study of waves in the quark-gluon fluid has been mostly performed with the assumption that the amplitude of the perturbations is small enough to justify the linearization of the Euler and continuity equations [11]. The analysis of perturbations with the linearized relativistic hydrodynamics leads to the standard second order wave equations and their traveling wave solutions, such as acoustic waves in the QGP. While linearization is justified in many cases, in others it should be replaced by another technique to treat perturbations keeping the nonlinearities of the theory. Since long ago there is a technique which preserves nonlinearities in the derivation of the differential equations which govern the evolution of perturbations. This is the reductive perturbation method (RPM) [16–19].

Nonlinearities may lead, as they do in other domains of physics, to new and interesting phenomena. In a pioneering work [20], with the use of nonrelativistic ideal hydrodynamics combined with the RPM and with an appropriate equation of state of cold nuclear matter, it was shown that it is possible to derive a Korteweg- de Vries (KdV) equation for the baryon density, which has analytic solitonic solutions. This suggests that a pulse in baryon density (the KdV soliton) can propagate without dissipation through the nuclear medium. If this occurs this pulse might be responsible for an interesting phenomena: the apparent nuclear transparency in proton nucleus collisions at low energies. The incoming proton would be absorbed by the fluid (target nucleus) and turned into a density pulse. In the fluid the pulse satisfies the KdV equation and it is able to traverse the target nucleus emerging on the other side. This can be called “nuclear transparency” and it is illustrated in the Fig 1.

Perturbations in fluids with different equations of state (EOS) generate different nonlinear wave equations: the breaking wave equation, KdV, Burgers... etc. Among these equations

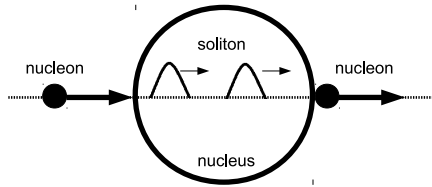


FIG. 1: KdV soliton in the baryon density simulating “nuclear transparency”.

we find the Kadomtsev-Petviashvili (KP) equation [21], which is a nonlinear wave equation in three spatial and one temporal coordinate. It is the generalization of the KdV equation to higher dimensions. This equation has been found with the application of the reductive perturbation method [16] to several different problems such as the propagation of solitons in multicomponent plasmas, dust acoustic waves in hot dust plasmas and dense electron-positron-ion plasma [22–34]. Previous studies on nonlinear waves in cold and warm nuclear matter can be found in [20, 35–40]. Works on nonlinear waves in cold QGP in the mean field approach were published in [41] and their extension to three dimensions was published in [42].

In this text we review the applications of the RPM [16, 22–33] to relativistic fluid dynamics [9, 10]. In the next section we review the basic equations of relativistic hydrodynamics. In section III we present the RPM and give the coordinate transformations to be used in the subsequent sections. In section IV we introduce the equations of state of hadronic matter and of the quark-gluon plasma, giving special attention to the ingredients which may generate solitons. In section V we list the wave equations which follow from the application of the

RPM to hydrodynamics. In section VI we present the analytical solutions of some of these differential equations and in section VII we show the time evolution of several nonlinear waves obtained by numerical integration. In section VIII we make some final remarks.

## II. RELATIVISTIC HYDRODYNAMICS

### A. Definitions and basic equations

As we have mentioned before, the hot and dense medium created in heavy ion collisions at RHIC behaves approximately as a perfect fluid and ideal hydrodynamics can be applied to describe its space-time evolution. Moreover the study of perturbations in the fluid, such as the waves created by fast partons, can be studied in the context of hydrodynamics as well. In this section we present the formalism of relativistic hydrodynamics. Pedagogical texts on this subject can be found in [9, 10]. Here we give special attention to the variables which are relevant for strongly interacting fluids, such as the baryon density. We write the final equations in a more extended form, which allows the reader to make a direct comparison with the corresponding non-relativistic versions of these equations. In what follows we use  $c = 1$ ,  $\hbar = 1$  and the Boltzmann constant is taken to be one, i.e.,  $k_B = 1$ . All the relativistic equations are written in terms of 4-vectors and the metric tensor is given by  $g_{\mu\nu}$ , with  $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$  and  $g_{\mu\nu} = 0$  if  $\mu \neq \nu$ . From [9, 10, 12] the fundamental equations of a relativistic ideal fluid are:

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu = 0 \quad (1)$$

and

$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p = 0 \quad (2)$$

where:

$$D \equiv u^\mu \partial_\mu \quad \text{and} \quad \nabla^\alpha \equiv \Delta^{\mu\alpha} \partial_\mu \quad (3)$$

and the projection operator on the orthogonal direction to the fluid velocity  $u^\mu$  is given by the tensor:

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \quad (4)$$

which has the properties  $\Delta^{\mu\nu}u_\mu = \Delta^{\mu\nu}u_\nu = 0$  and  $\Delta^{\mu\nu}\Delta_\nu^\alpha = \Delta^{\mu\alpha}$ .

The velocity 4-vector of the fluid element is given by [9, 10, 12]:  $u^\mu = (\gamma, \gamma \vec{v})$  where  $\gamma$  is the Lorentz factor  $\gamma = (1 - v^2)^{-1/2}$  and thus  $u^\mu u_\mu = 1$  and  $u_\mu \partial_\nu u^\mu = (1/2) \partial_\nu (u^\mu u_\mu) = (1/2) \partial_\nu (1) = 0$ .

In an ideal fluid all dissipative (viscous) effects are neglected and in order to introduce the effects of viscosity, it is necessary to add the viscous stress tensor  $\Pi^{\mu\nu}$  to the energy-momentum tensor. In a simple way we may write for a viscous fluid:

$$T^{\mu\nu} = T_{(0)}^{\mu\nu} + \Pi^{\mu\nu} \quad (5)$$

where

$$T_{(0)}^{\mu\nu} = \varepsilon u^\mu u^\nu - p \Delta^{\mu\nu} \quad (6)$$

is the ideal relativistic fluid energy-momentum tensor [9, 10]. We also consider for simplicity [12] a system without conserved charges (or at zero chemical potential) and so the total momentum density is due to the flow of energy density  $u_\mu T^{\mu\nu} = \varepsilon u^\nu$  and hence we must assume that:

$$u_\mu \Pi^{\mu\nu} = 0 \quad (7)$$

We take the appropriate projections of the conservation equations of the energy momentum tensor: the parallel ( $u_\nu \partial_\mu T^{\mu\nu}$ ) and perpendicular ( $\Delta_\nu^\alpha \partial_\mu T^{\mu\nu}$ ) to the fluid velocity. The results are:

$$u_\nu \partial_\mu T^{\mu\nu} = D\varepsilon + (\varepsilon + p) \partial_\mu u^\mu + u_\nu \partial_\mu \Pi^{\mu\nu} = 0 \quad (8)$$

and

$$\Delta_\nu^\alpha \partial_\mu T^{\mu\nu} = (\varepsilon + p) D u^\alpha - \nabla^\alpha p + \Delta_\nu^\alpha \partial_\mu \Pi^{\mu\nu} = 0 \quad (9)$$

Using the symmetrization notation:

$$A_{(\mu} B_{\nu)} = \frac{1}{2} (A_\mu B_\nu + A_\nu B_\mu) \quad (10)$$

we are able to rewrite the  $u_\nu \partial_\mu \Pi^{\mu\nu}$  term in (8) as  $u_\nu \partial_\mu \Pi^{\mu\nu} = \partial_\mu (u_\nu \Pi^{\mu\nu}) - \Pi^{\mu\nu} \partial_{(\mu} u_{\nu)}$ . We also use the identity

$$\partial_\mu = u_\mu D + \nabla_\mu \quad (11)$$

and the choice of frame  $u_\mu \Pi^{\mu\nu} = 0$ . The fundamental equations of relativistic viscous fluid dynamics are finally given by:

$$D\varepsilon + (\varepsilon + p) \partial_\mu u^\mu - \Pi^{\mu\nu} \nabla_{(\mu} u_{\nu)} = 0 \quad (12)$$

and:

$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta_\nu^\alpha \partial_\mu \Pi^{\mu\nu} = 0 \quad (13)$$

## B. The viscous stress tensor

The viscous stress tensor  $\Pi^{\mu\nu}$  has not been specified yet. We shall derive it with the help of the second law of thermodynamics, which states that entropy density must always increase locally. Moreover we shall consider a system in thermodynamic equilibrium ( $dp = 0$ ) with zero chemical potential. Based on these statements, the thermodynamic relations are given by:

$$\varepsilon + p = Ts \quad \text{and} \quad Tds = d\varepsilon \quad (14)$$

The second law of thermodynamics is considered in the covariant form:

$$\partial_\mu s^\mu \geq 0 \quad (15)$$

where the 4-current  $s^\mu$  is given by:

$$s^\mu = su^\mu \quad (16)$$

Inserting relations (14) into the second law (15) and assuming that  $Dp = 0$  we find:

$$\partial_\mu s^\mu = Ds + s\partial_\mu u^\mu = \frac{D\varepsilon}{T} + \frac{(\varepsilon + p)}{T}\partial_\mu u^\mu \quad (17)$$

Using now equation (12) we obtain:

$$\partial_\mu s^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} \geq 0 \quad (18)$$

The viscous tensor may be decomposed into a traceless part  $\pi^{\mu\nu}$  ( $\pi_\mu^\mu = 0$ ) and a non-vanishing trace part, so that:

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Pi \quad (19)$$

In terms of the notation introduced in [12]:

$$\nabla_{\langle\mu}u_{\nu\rangle} \equiv 2\nabla_{(\mu}u_{\nu)} - \frac{2}{3}\Delta_{\mu\nu}\nabla_\alpha u^\alpha \quad (20)$$

it can be shown that:

$$u_\mu \pi^{\mu\nu} = 0 \quad (21)$$

$$u^\mu u^\nu \nabla_{\langle\mu}u_{\nu\rangle} = 0 \quad (22)$$

$$g^{\mu\nu}\nabla_{\langle\mu}u_{\nu\rangle} = 0 \quad (23)$$

and

$$\Delta^{\mu\nu}\Delta_{\mu\nu} = 3 \quad (24)$$

Inserting (19) to (24) into (18) we find:

$$\partial_\mu s^\mu = \frac{1}{2T}\pi^{\mu\nu}\nabla_{\langle\mu}u_{\nu\rangle} + \frac{1}{T}\Pi\nabla_\alpha u^\alpha \geq 0 \quad (25)$$

and this inequality (a positive sum of squares) is satisfied if:

$$\pi^{\mu\nu} = \eta\nabla^{\langle\mu}u^{\nu\rangle}, \quad \Pi = \zeta\nabla_\alpha u^\alpha, \quad \eta \geq 0 \quad \text{and} \quad \zeta \geq 0 \quad (26)$$

where  $\eta$  is the *shear viscosity coefficient* and  $\zeta$  is the *bulk viscosity coefficient*. The final expression for the viscous tensor is given by the substitution of (26) in (19):

$$\Pi^{\mu\nu} = \eta\nabla^{\langle\mu}u^{\nu\rangle} + \zeta\Delta^{\mu\nu}\nabla_\alpha u^\alpha \quad (27)$$

### C. The relativistic Navier-Stokes equation

The system of equations (12), (13) and (27) is called relativistic Navier-Stokes equation. The temporal component ( $\alpha = 0$ ) of (13) multiplied by  $v^i$  is given by:

$$(\varepsilon + p)(D\gamma)v^i - v^i\nabla^0 p + v^i\Delta_\nu^0\partial_\mu\Pi^{\mu\nu} = 0 \quad (28)$$

and the spatial component ( $\alpha = i$ ) of (13):

$$(\varepsilon + p)(D\gamma)v^i = -(\varepsilon + p)\gamma(Dv^i) + \nabla^i p - \Delta_\nu^i\partial_\mu\Pi^{\mu\nu} \quad (29)$$

Inserting (29) into (28) we find:

$$(\varepsilon + p)\gamma(Dv^i) + (v^i\nabla^0 - \nabla^i)p - (v^i\Delta_\nu^0 - \Delta_\nu^i)\partial_\mu\Pi^{\mu\nu} = 0 \quad (30)$$

which can be rewritten with the help of (3) and (4) as:

$$(\varepsilon + p)\gamma^2\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right)v^i + v^i\frac{\partial p}{\partial t} - \partial^i p - (v^i\Delta_\nu^0 - \Delta_\nu^i)\partial_\mu\Pi^{\mu\nu} = 0 \quad (31)$$

The ideal relativistic fluid limit  $\Pi^{\mu\nu} = 0$  ( $\eta = \zeta = 0$ ) in the last equation provides the relativistic version of the Euler equation [9, 10, 12]. For future use we rewrite (31) in detail.

To do so, we recall (3), (4) and (27) to obtain the expression for  $\partial_\mu\Pi^{\mu\nu}$  in (31):

$$\partial_\mu\Pi^{\mu\nu} = \eta\left\{\partial_\mu\partial^\mu u^\nu + \partial_\mu\partial^\nu u^\mu - \partial_\mu\left[\gamma\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right)(u^\mu u^\nu)\right]\right\}$$



$$+ \left( \zeta - \frac{2}{3}\eta \right) \partial^\nu \left[ \frac{\partial \gamma}{\partial t} + \vec{\nabla} \cdot (\gamma \vec{v}) \right] - \left( \zeta - \frac{2}{3}\eta \right) \partial_\mu \left\{ u^\mu u^\nu \left[ \frac{\partial \gamma}{\partial t} + \vec{\nabla} \cdot (\gamma \vec{v}) \right] \right\} \quad (32)$$

and then insert this last result into (31). We find after some algebra [43]:

$$\begin{aligned} & (\varepsilon + p) \gamma^2 \left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \vec{v} + \vec{v} \frac{\partial p}{\partial t} + \vec{\nabla} p \\ & - \eta \vec{v} \left\{ \partial_\mu \partial^\mu \gamma + \partial_\mu \frac{\partial u^\mu}{\partial t} - \partial_\mu \left[ \gamma \left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) (\gamma u^\mu) \right] \right\} - \vec{v} \left( \zeta - \frac{2}{3}\eta \right) \frac{\partial}{\partial t} \left[ \frac{\partial \gamma}{\partial t} + \vec{\nabla} \cdot (\gamma \vec{v}) \right] \\ & + \vec{v} \left( \zeta - \frac{2}{3}\eta \right) \partial_\mu \left\{ \gamma u^\mu \left[ \frac{\partial \gamma}{\partial t} + \vec{\nabla} \cdot (\gamma \vec{v}) \right] \right\} \\ & + \eta \left\{ \partial_\mu \partial^\mu (\gamma \vec{v}) - \partial_\mu \vec{\nabla} u^\mu - \partial_\mu \left[ \gamma \left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) (\gamma \vec{v} u^\mu) \right] \right\} - \left( \zeta - \frac{2}{3}\eta \right) \vec{\nabla} \left[ \frac{\partial \gamma}{\partial t} + \vec{\nabla} \cdot (\gamma \vec{v}) \right] \\ & - \left( \zeta - \frac{2}{3}\eta \right) \partial_\mu \left\{ \gamma \vec{v} u^\mu \left[ \frac{\partial \gamma}{\partial t} + \vec{\nabla} \cdot (\gamma \vec{v}) \right] \right\} = 0 \end{aligned} \quad (33)$$

which is the relativistic version of the Navier-Stokes equation. The perfect fluid, described by  $\eta = \zeta = 0$  in (33) gives the relativistic version of Euler equation [9, 10]:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{1}{(\varepsilon + p) \gamma^2} \left( \vec{\nabla} p + \vec{v} \frac{\partial p}{\partial t} \right) \quad (34)$$

#### D. Causality and the relativistic Navier-Stokes equation

The relativistic Navier-Stokes equation (33) does not constitute a causal theory [12]. This fact can be understood when small perturbations are considered in a system in equilibrium with energy density  $\varepsilon_0$ , pressure  $p_0$  and at rest ( $\vec{v} = \vec{0}$ ).

Such perturbations can be described by:

$$\varepsilon = \varepsilon_0 + \delta \varepsilon(t, \vec{x}) , \quad p = p_0 + \delta p(t, \vec{x}) \quad \text{and} \quad u^\mu = (1, \vec{0}) + \delta u^\mu(t, \vec{x}) \quad (35)$$

where “ $\delta$ ” denotes small deviation from equilibrium. For simplicity we consider perturbations which depend only on the  $x$  space coordinate. For the particular direction  $\alpha = y$  we insert (35) into (13) to find:

$$(\varepsilon_0 + \delta \varepsilon + p_0 + \delta p) D(\delta u^y) - \nabla^y (p_0 + \delta p) + \Delta_\nu^y \partial_\mu \Pi^{\mu\nu} = 0$$

Considering only the dependence on the  $x$  coordinate it becomes:

$$(\varepsilon_0 + p_0) \frac{\partial}{\partial t} \delta u^y + \frac{\partial}{\partial x} \Pi^{xy} + \mathcal{O}(\delta^2) = 0 \quad (36)$$

Analogously we insert (35) into the viscous tensor (19) for the particular direction  $\nu = y$  and considering only the  $x$  coordinate dependence:

$$\Pi^{xy} = -\eta \frac{\partial}{\partial x} \delta u^y + \mathcal{O}(\delta^2) \quad (37)$$

Substituting (37) in (36) we find:

$$(\varepsilon_0 + p_0) \frac{\partial}{\partial t} \delta u^y - \eta \frac{\partial^2}{\partial x^2} \delta u^y = \mathcal{O}(\delta^2) \quad (38)$$

which provides, after performing the linearization approximation (i.e., neglecting  $\mathcal{O}(\delta^2)$ ) the diffusion-type evolution equation:

$$\frac{\partial}{\partial t} \delta u^y - \frac{\eta}{(\varepsilon_0 + p_0)} \frac{\partial^2}{\partial x^2} \delta u^y = 0 \quad (39)$$

for the perturbation  $\delta u^y = \delta u^y(t, x)$ .

The acausality property of the Navier-Stokes equation can be studied from the Laplace-Fourier wave ansatz for  $\delta u^y$ :

$$\delta u^y(t, x) = f_{(w,k)} e^{ikx - wt} \quad (40)$$

where  $f_{(w,k)}$  are the coefficient for a given frequency  $w$  and a given wave number  $k$ . When inserting (40) in (143) we obtain the dispersion-relation of the diffusion equation:

$$w = \frac{\eta}{(\varepsilon_0 + p_0)} k^2 \quad (41)$$

which gives the speed of diffusion of a mode with wavenumber  $k$  :

$$V_{diff}(k) = \frac{dw}{dk} = 2 \frac{\eta}{(\varepsilon_0 + p_0)} k \quad (42)$$

The causality violation occurs when

$$\lim_{k \rightarrow \infty} V_{diff}(k) \rightarrow \infty \quad (43)$$

which exceeds the speed of light, i.e , the speed of diffusion grows without bound for sufficiently large wavenumber due its linear dependence on the wavenumber.

A possible way to regulate the viscous fluid theory is given by the “Maxwell-Cattaneo law”. In this approach a new transport coefficient called relaxation time ( $\tau_\pi$ ) is added to equation (37) as follows:

$$\tau_\pi \frac{\partial}{\partial t} \Pi^{xy} + \Pi^{xy} = -\eta \frac{\partial}{\partial x} \delta u^y + \mathcal{O}(\delta^2) \quad (44)$$

The derivative of the above equation with respect to  $x$  provides:

$$\tau_\pi \frac{\partial}{\partial t} \frac{\partial}{\partial x} \Pi^{xy} + \frac{\partial}{\partial x} \Pi^{xy} = -\eta \frac{\partial^2}{\partial x^2} \delta u^y + \mathcal{O}(\delta^2) \quad (45)$$

which becomes, after substituting  $\frac{\partial}{\partial x} \Pi^{xy}$  from (36) and performing the linearization approximation, the following evolution equation:

$$\tau_\pi \frac{\partial^2}{\partial t^2} \delta u^y + \frac{\partial}{\partial t} \delta u^y - \frac{\eta}{(\varepsilon_0 + p_0)} \frac{\partial^2}{\partial x^2} \delta u^y = 0 \quad (46)$$

where the term with second derivative with respect to time provides the following dispersion relation when (40) is used:

$$w = \frac{1}{2\tau_\pi} \pm \sqrt{\frac{1}{4\tau_\pi^2} - \frac{\eta k^2}{(\varepsilon_0 + p_0)\tau_\pi}} = \frac{1}{2\tau_\pi} \pm ik \sqrt{\frac{\eta}{(\varepsilon_0 + p_0)\tau_\pi} - \frac{1}{4\tau_\pi^2 k^2}}$$

$$|w|^2 = w\bar{w} = k^2 \frac{\eta}{(\varepsilon_0 + p_0)\tau_\pi} \quad (47)$$

The speed of diffusion of a mode with wavenumber  $k$  is now:

$$V_{diff_{causal}}(k) = \frac{d|w|}{dk} = \sqrt{\frac{\eta}{(\varepsilon_0 + p_0)\tau_\pi}} \quad (48)$$

which does not violate causality:

$$\lim_{k \rightarrow \infty} V_{diff}(k) = \sqrt{\frac{\eta}{(\varepsilon_0 + p_0)\tau_\pi}} \quad (49)$$

where  $\tau_\pi \neq 0$ . Beyond the Maxwell-Cattaneo law, the more complete formulation of viscous hydrodynamics is the Müller-Israel-Stewart theory [12], which contains the Maxwell-Cattaneo law as a limit. A more precise study is found in [12]. Here we just wish to point out that there are alternative approaches, in which causality is preserved.

We present a different approximation scheme, which goes beyond the linear approximation, preserves nonlinear terms and does not violate causality: the reductive perturbation method, a technique that will be presented in the next section. When this method is applied to the relativistic Navier-Stokes equation (33) we arrive at the conclusion that, for the purpose of studying perturbations, the equation:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{(\varepsilon + p)} \left[ \vec{\nabla} p + \vec{v} \frac{\partial p}{\partial t} \right] + \frac{1}{(\varepsilon + p)} \left[ \eta \vec{\nabla}^2 \vec{v} + \left( \zeta + \frac{1}{3} \eta \right) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \right] \quad (50)$$

is equivalent to (33). In other words, considering (50) or (33) leads to the same wave equation.

## E. Continuity equations

### 1. Entropy

The continuity equation for the entropy density is given by (25):

$$\partial_\mu s^\mu = \frac{1}{2T} \eta \nabla^{\langle\mu} u^{\nu\rangle} \nabla_{\langle\mu} u_{\nu\rangle} + \frac{1}{T} \zeta (\nabla_\alpha u^\alpha)^2 \quad (51)$$

Using (3), (4),  $u_\nu \partial_\mu u^\nu = 0$  and  $u^\nu u_\nu = 1$  we can rewrite the above equation in the form:

$$\partial_\mu s^\mu = -\frac{\eta}{T} (\partial^\mu u^\nu) \partial_\nu u_\mu + \frac{1}{T} \left( \frac{2}{3} \eta + \zeta \right) (\partial_\mu u^\mu)^2 \quad (52)$$

Using (16) and  $u^\mu = (\gamma, \gamma \vec{v})$  the last equation can also be rewritten as:

$$\begin{aligned} \gamma \frac{\partial s}{\partial t} + \gamma \vec{\nabla} s \cdot \vec{v} + s \frac{\partial \gamma}{\partial t} + s \vec{\nabla} \gamma \cdot \vec{v} + \gamma s \vec{\nabla} \cdot \vec{v} = & -\frac{\eta}{T} \left( \frac{\partial \gamma}{\partial t} \right)^2 - 2 \frac{\eta}{T} \left[ \vec{\nabla} \gamma \cdot \frac{\partial}{\partial t} (\gamma \vec{v}) \right] \\ & - \frac{\eta}{T} (\partial^i u^j) \partial_j u_i + \frac{1}{T} \left( \frac{2}{3} \eta + \zeta \right) \left[ \frac{\partial \gamma}{\partial t} + \gamma \vec{\nabla} \cdot \vec{v} + \vec{\nabla} \gamma \cdot \vec{v} \right]^2 \end{aligned} \quad (53)$$

which is the relativistic version of the continuity equation for the entropy density  $s$ . In the case of an ideal fluid ( $\eta = \zeta = 0$ ) we recover the entropy density conservation:

$$\gamma \frac{\partial s}{\partial t} + \gamma \vec{\nabla} s \cdot \vec{v} + s \frac{\partial \gamma}{\partial t} + s \vec{\nabla} \gamma \cdot \vec{v} + \gamma s \vec{\nabla} \cdot \vec{v} = 0$$

which, with the use of:

$$\frac{\partial \gamma}{\partial t} = \gamma^3 v \frac{\partial v}{\partial t} \quad \text{and} \quad \vec{\nabla} \gamma = \gamma^3 v \vec{\nabla} v \quad (54)$$

becomes [9, 10]:

$$\frac{\partial s}{\partial t} + \gamma^2 v s \left( \frac{\partial v}{\partial t} + \vec{v} \cdot \vec{\nabla} v \right) + \vec{\nabla} \cdot (s \vec{v}) = 0 \quad (55)$$

### 2. Baryon density

The relativistic version of the continuity equation for the baryon density  $\rho_B$  in ideal relativistic hydrodynamics is:

$$\partial_\nu j_B^\nu = 0 \quad (56)$$

Since  $j_B^\nu = u^\nu \rho_B$  and using (54) the last equation can be rewritten as [9]:

$$\frac{\partial \rho_B}{\partial t} + \gamma^2 v \rho_B \left( \frac{\partial v}{\partial t} + \vec{v} \cdot \vec{\nabla} v \right) + \vec{\nabla} \cdot (\rho_B \vec{v}) = 0 \quad (57)$$

## F. Non-relativistic limit

In what follows we recover the non-relativistic limit of the continuity equation for the entropy density, continuity equation for the baryon density and also for the Navier-Stokes equation. The non-relativistic limit is essentially given by  $v^2 \ll 1$  and  $\gamma \cong 1$ .

### 1. Continuity equation

Using  $v^2 \ll 1$  and  $\gamma \cong 1$  in the equation for entropy density (53) we find:

$$\frac{\partial s}{\partial t} + \vec{\nabla} \cdot (s\vec{v}) = -\frac{\eta}{T}(\partial^i u^j)\partial_j u_i + \frac{1}{T}\left(\frac{2}{3}\eta + \zeta\right)(\vec{\nabla} \cdot \vec{v})^2 \quad (58)$$

For a perfect fluid  $\eta = \zeta = 0$  we obtain the usual continuity equation [9]:

$$\frac{\partial s}{\partial t} + \vec{\nabla} \cdot (s\vec{v}) = 0$$

For the baryon density continuity equation we have, after applying  $v^2 \ll 1$  and  $\gamma \cong 1$  in (57):

$$\frac{\partial \rho_B}{\partial t} + \vec{\nabla} \cdot (\rho_B \vec{v}) = 0 \quad (59)$$

### 2. Navier-Stokes equation

The energy density of the fluid element of mass  $M$  and momentum  $P$  is:

$$\varepsilon = \frac{E}{Vol} = \frac{\sqrt{M^2 + P^2}}{Vol}$$

In the non-relativistic limit:  $M \gg P$  we have:

$$\varepsilon \cong \frac{\sqrt{M^2}}{Vol} = \frac{M}{Vol}$$

The volumetric density of fluid matter is given by  $\frac{M}{Vol} = \rho$  and hence:

$$\varepsilon \cong \rho \quad (60)$$

The pressure is defined as the ratio of the force per area of the fluid element  $A$ . The force is given by the time derivative of the momentum  $P$ . The pressure is then:

$$p = \frac{\text{force}}{A} = \frac{1}{A} \frac{dP}{dt} \quad (61)$$

and so

$$\vec{\nabla}p + \vec{v}\frac{\partial p}{\partial t} = \frac{1}{A}\vec{\nabla}\frac{dP}{dt} + \frac{1}{A}\vec{v}\frac{d^2P}{dt^2}$$

For  $v^2 \ll 1$  we find:

$$|\vec{v}|\frac{d^2P}{dt^2} \ll \left|\vec{\nabla}\frac{dP}{dt}\right|$$

and consequently:

$$\vec{\nabla}p + \vec{v}\frac{\partial p}{\partial t} \cong \frac{1}{A}\vec{\nabla}\frac{dP}{dt}$$

With the use of (61) we finally obtain:

$$\vec{\nabla}p + \vec{v}\frac{\partial p}{\partial t} \cong \vec{\nabla}p \quad (62)$$

According to these last approximations and using (60) we conclude that  $\varepsilon + p$  takes the form:

$$\varepsilon + p \cong \rho + \frac{1}{A}\frac{dP}{dt}$$

and with  $\rho \gg \frac{1}{A}\frac{dP}{dt}$  (because  $M \gg P$ ) we find:

$$\varepsilon + p \cong \rho \quad (63)$$

Inserting  $\gamma \cong 1$ , (62) and (63) in (33) we find:

$$\begin{aligned} & \rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right] + \vec{\nabla}p \\ & \eta \left\{ -\vec{\nabla}^2 \vec{v} - \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \left( \frac{\partial \vec{v}}{\partial t} \cdot \vec{\nabla} \right) \vec{v} - (\vec{v} \cdot \vec{\nabla}) \frac{\partial \vec{v}}{\partial t} - \left( \vec{\nabla} \cdot \frac{\partial \vec{v}}{\partial t} \right) \vec{v} \right. \\ & \quad \left. - \vec{\nabla} \cdot \vec{v} \frac{\partial \vec{v}}{\partial t} - \vec{v} \cdot \vec{\nabla} [(\vec{v} \cdot \vec{\nabla})\vec{v}] \right\} \\ & + \left( \zeta - \frac{2}{3}\eta \right) \left\{ -\vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \frac{\partial \vec{v}}{\partial t} \cdot \vec{\nabla} - \vec{v} \cdot \vec{\nabla} \cdot \frac{\partial \vec{v}}{\partial t} - [(\vec{v} \cdot \vec{\nabla})\vec{v}] \vec{\nabla} \cdot \vec{v} \right\} = 0 \end{aligned} \quad (64)$$

Neglecting the terms of order  $v^3$ , which are  $\vec{v} \cdot \vec{\nabla} [(\vec{v} \cdot \vec{\nabla})\vec{v}]$  and  $[(\vec{v} \cdot \vec{\nabla})\vec{v}] \vec{\nabla} \cdot \vec{v}$  we have:

$$\begin{aligned} & \rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right] + \vec{\nabla}p - \eta \vec{\nabla}^2 \vec{v} - \left( \zeta + \frac{1}{3}\eta \right) \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) \\ & - \left( \zeta + \frac{1}{3}\eta \right) \frac{\partial}{\partial t} [\vec{v}(\vec{\nabla} \cdot \vec{v})] - \eta \frac{\partial}{\partial t} [(\vec{v} \cdot \vec{\nabla})\vec{v}] = 0 \end{aligned} \quad (65)$$

We still need to estimate the relative size of the terms in last equation. This is made by the comparison of terms without viscosity to others with viscosity. From (65):

$$\rho(\vec{v} \cdot \vec{\nabla})\vec{v} - \eta \frac{\partial}{\partial t} [(\vec{v} \cdot \vec{\nabla})\vec{v}] = \frac{M}{Vol}(\vec{v} \cdot \vec{\nabla})\vec{v} - \frac{\eta}{M^2} \frac{\partial}{\partial t} [(\vec{P} \cdot \vec{\nabla})\vec{P}]$$

where we have used the volumetric density of the fluid matter  $\frac{M}{Vol} = \rho$  and the non-relativistic momentum  $\vec{P} = M\vec{v}$  for the element with mass  $M$  for the fluid matter. In the non-relativistic limit we have, as usual:  $M \gg P$  and the last expression becomes:

$$\rho(\vec{v} \cdot \vec{\nabla})\vec{v} - \eta \frac{\partial}{\partial t}[(\vec{v} \cdot \vec{\nabla})\vec{v}] \cong \rho(\vec{v} \cdot \vec{\nabla})\vec{v} \quad (66)$$

since

$$\frac{M}{Vol}(\vec{v} \cdot \vec{\nabla})\vec{v} \gg -\frac{\eta}{M^2} \frac{\partial}{\partial t}[(\vec{P} \cdot \vec{\nabla})\vec{P}]$$

Analogously we also have from (65):

$$\rho \frac{\partial \vec{v}}{\partial t} - \left(\zeta + \frac{1}{3}\eta\right) \frac{\partial}{\partial t}[\vec{v}(\vec{\nabla} \cdot \vec{v})] \cong \rho \frac{\partial \vec{v}}{\partial t} \quad (67)$$

since

$$\frac{M}{Vol} \frac{\partial \vec{v}}{\partial t} \gg -\left(\zeta + \frac{1}{3}\eta\right) \frac{1}{M^2} \frac{\partial}{\partial t}[\vec{P}(\vec{\nabla} \cdot \vec{P})]$$

The non-relativistic Navier-Stokes equation is then given by using (66) and (67) in (65):

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho} \vec{\nabla} p + \frac{\eta}{\rho} \vec{\nabla}^2 \vec{v} + \frac{1}{\rho} \left(\zeta + \frac{1}{3}\eta\right) \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) \quad (68)$$

Again, the perfect fluid is described by  $\eta = \zeta = 0$  in (68) and gives the non-relativistic version of Euler equation [9, 10]:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho} \vec{\nabla} p \quad (69)$$

### III. THE REDUCTIVE PERTURBATION METHOD

#### A. Linearization

We start from the equations of ideal relativistic hydrodynamics and, using the linearization approximation, we derive a wave equation for perturbations in the pressure. This equation has traveling wave solutions which represent acoustic waves. In the derivation presented here we follow closely the reference [11]. In the presence of perturbations the energy density and pressure for the relativistic fluid are written as (35):

$$\varepsilon(\vec{r}, t) = \varepsilon_0 + \delta\varepsilon(\vec{r}, t) \quad (70)$$

and

$$p(\vec{r}, t) = p_0 + \delta p(\vec{r}, t) \quad (71)$$

respectively. The uniform relativistic fluid is defined by  $\varepsilon_0$  and  $p_0$ , while  $\delta\varepsilon$  and  $\delta p$  correspond to perturbations in this fluid. Energy-momentum conservation implies that for an ideal fluid  $\Pi^{\mu\nu} = 0$  and from (5) we have:

$$\partial_\mu T^{\mu\nu} = 0 \quad (72)$$

where  $T^{\mu\nu}$  is the energy-momentum tensor given by:

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu} \quad (73)$$

Linearization consists in keeping only first order terms such as  $\delta\varepsilon$ ,  $\delta P$  and  $\vec{v}$  and neglect terms proportional to:

$$v^2, \quad v\delta\varepsilon, \quad v\delta P, \quad \vec{v} \cdot \vec{\nabla} v, \quad (\vec{v} \cdot \vec{\nabla})\vec{v} \quad (74)$$

and also neglect higher powers of these products or other combinations of them. Naturally we have  $\gamma \sim 1$ . From (72) we have:

$$u^\mu \partial_\nu [(\varepsilon + p)u^\nu] + (\varepsilon + p)u^\nu \partial_\nu u^\mu - \partial_\nu (pg^{\nu\mu}) = 0 \quad (75)$$

The temporal component ( $\mu = 0$ ) of the above equation is given by:

$$\gamma \partial_0 [(\varepsilon + p)\gamma] + \gamma \partial_i [(\varepsilon + p)u^i] + (\varepsilon + p)u^0 \partial_0 \gamma + (\varepsilon + p)u^i \partial_i \gamma - \partial_0 p = 0 \quad (76)$$

which, after using (74) and  $\gamma \sim 1$ , becomes:

$$\partial_0(\varepsilon + p) + \partial_i [(\varepsilon + p)v^i] - \partial_0 p = 0$$

or

$$\frac{\partial \varepsilon}{\partial t} + \vec{\nabla} \cdot [(\varepsilon + p)\vec{v}] = 0 \quad (77)$$

For the  $j$ -th spatial component ( $\mu = j$ ) in (75) we have:

$$u^j \partial_0 [(\varepsilon + p)u^0] + u^j \partial_i [(\varepsilon + p)u^i] + (\varepsilon + p)u^0 \partial_0 u^j + (\varepsilon + p)u^i \partial_i u^j - \partial^j p = 0$$

which, with the use of (74) and  $\gamma \sim 1$ , becomes:

$$\frac{\partial}{\partial t} [(\varepsilon + p)\vec{v}] + \vec{\nabla} p = 0 \quad (78)$$

Substituting the expansions (70) and (71) in (77) and (78) we find:

$$\frac{\partial}{\partial t} [\varepsilon_0 + \delta\varepsilon] + \vec{\nabla} \cdot [(\varepsilon_0 + \delta\varepsilon + p_0 + \delta p)\vec{v}] = 0 \quad (79)$$



and

$$\frac{\partial}{\partial t}[(\varepsilon_0 + \delta\varepsilon + p_0 + \delta p)\vec{v}] + \vec{\nabla}[p_0 + \delta p] = 0 \quad (80)$$

Neglecting the terms listed in (74) in (79) and (80) these equations become:

$$\frac{\partial(\delta\varepsilon)}{\partial t} + (\varepsilon_0 + p_0)\vec{\nabla} \cdot \vec{v} = 0 \quad (81)$$

and

$$(\varepsilon_0 + p_0)\frac{\partial\vec{v}}{\partial t} + \vec{\nabla}(\delta p) = 0 \quad (82)$$

Equation (81) expresses energy conservation and equation (82) is Newton's second law. Integrating (82) with respect to the time and setting the integration constant to zero we find:

$$\vec{v} = -\frac{1}{(\varepsilon_0 + p_0)} \int \vec{\nabla}(\delta p) dt \quad (83)$$

which inserted in (81) yields:

$$\frac{\partial(\delta\varepsilon)}{\partial t} - \int \vec{\nabla}^2(\delta p) dt = 0 \quad (84)$$

Performing the time derivative we obtain:

$$\frac{\partial^2(\delta\varepsilon)}{\partial t^2} - \vec{\nabla}^2(\delta p) = 0 \quad (85)$$

Assuming that

$$\delta\varepsilon = \frac{\partial\varepsilon}{\partial p}\delta p \quad (86)$$

with  $\partial\varepsilon/\partial p$  being a constant, we have (85) rewritten as:

$$\frac{\partial\varepsilon}{\partial p} \frac{\partial^2(\delta p)}{\partial t^2} - \vec{\nabla}^2(\delta p) = 0 \quad (87)$$

The above expression is a wave equation from where we can identify the velocity of propagation as:

$$c_s = \left(\frac{\partial p}{\partial\varepsilon}\right)^{1/2} \quad (88)$$

where  $c_s$  is the speed of sound. Equation (87) can then be finally written as:

$$\vec{\nabla}^2(\delta p) - \frac{1}{c_s^2} \frac{\partial^2(\delta p)}{\partial t^2} = 0 \quad (89)$$

which describes the propagation of a pressure wave in the fluid. This study ensures that the analysis of perturbations with the linearized relativistic hydrodynamics leads to the standard second order linear wave equations and their traveling wave solutions, such as acoustic waves in the hadronic medium.

## B. Beyond linearization

While linearization is justified in many cases, in others, where perturbations are not so small, it should be replaced by another technique to treat perturbations keeping the nonlinearities of the theory. This is where a physical theory, in our case relativistic hydrodynamics, may benefit from developments in applied mathematics. Indeed, since long ago there is a technique which preserves nonlinearities in the derivation of the differential equations which govern the evolution of perturbations. This is the reductive perturbation method (RPM) [16–19].

We start the RPM description by considering the simple linear wave equation:

$$\frac{\partial F}{\partial t} + \alpha \frac{\partial F}{\partial x} = 0 \quad (90)$$

which describes one-dimensional waves in an ideal fluid. In the above equation  $\alpha$  is a constant. Let us expand of  $F(x, t)$  around the constant value  $F_0$  in terms of the small expansion parameter  $\sigma$  ( $0 < \sigma < 1$ ) :

$$F(x, t) = F_0 + \sigma F_1(x, t) + \sigma^2 F_2(x, t) + \sigma^3 F_3(x, t) + \dots \quad (91)$$

Inserting (91) into (90) we obtain a series:

$$\sum_j \sigma^j \left\{ \frac{\partial F_j}{\partial t} + \alpha \frac{\partial F_j}{\partial x} \right\} = 0 \quad \text{for } j = 1, 2, 3, \dots \quad (92)$$

The coefficients of each power of  $\sigma$  must vanish independently, i.e., each bracket must vanish. This condition yields a set of differential equations for the  $F_j$ :

$$\frac{\partial F_j}{\partial t} + \alpha \frac{\partial F_j}{\partial x} = 0 \quad (93)$$

Now we consider the simplest nonlinear wave equation, the so called breaking wave equation:

$$\frac{\partial F}{\partial t} + \alpha F \frac{\partial F}{\partial x} = 0 \quad (94)$$

where  $\alpha$  is a real coefficient. Performing the expansion (91) the above equation becomes:

$$\sigma \frac{\partial F_1}{\partial t} + \sigma^2 \frac{\partial F_2}{\partial t} + \sigma \alpha F_0 \frac{\partial F_1}{\partial x} + \sigma^2 \alpha F_0 \frac{\partial F_2}{\partial x} + \sigma^2 \alpha F_1 \frac{\partial F_1}{\partial x} + \dots = 0 \quad (95)$$

Let us look for simple nonlinear differential equations for the perturbations  $F_1, F_2, \dots$ . We expect to find algebraic structures similar to (94) in equation (95) in order  $\sigma$  and  $\sigma^2$ . To

do so, we introduce the new coordinates  $\tau$  and  $\xi$ , which are connected to  $t$  and  $x$ . This coordinate transformation is such that:

$$\frac{\partial}{\partial t} = \sigma^n \frac{\partial}{\partial \tau} \quad \text{and} \quad \frac{\partial}{\partial x} = \sigma^m \frac{\partial}{\partial \xi} \quad (96)$$

where  $n$  and  $m$  will be determined. We note that a highly nontrivial aspect of this transformation is that it contains the same small parameter  $\sigma$  used in the expansion (95). Inserting (96) into (95) we find the following equation:

$$\sigma^{n+1} \frac{\partial F_1}{\partial \tau} + \sigma^{n+2} \frac{\partial F_2}{\partial \tau} + \sigma^{m+2} \alpha F_1 \frac{\partial F_1}{\partial \xi} + \sigma \alpha F_0 \left[ \sigma^m \frac{\partial F_1}{\partial \xi} + \sigma^{m+1} \frac{\partial F_2}{\partial \xi} \right] = 0 \quad (97)$$

where, for the sake of simplicity, we have chosen  $F_0 = 0$  and we have truncated the sum keeping only the lowest order terms in  $\sigma$ . We observe that if we choose:

$$n + 1 = m + 2 \quad \text{and} \quad n + 2 \geq 3 \quad (98)$$

the first and third terms can be grouped together and, for the particular choice  $n = 3/2$  and  $m = 1/2$ , equation (97) becomes:

$$\sigma^2 \left[ \frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} \right] + \sigma^3 \left[ \frac{\partial F_2}{\partial \tau} \right] = 0 \quad (99)$$

where, just before arriving at the above expression, we have divided the whole equation by  $\sigma^{1/2}$ . From the term proportional to  $\sigma^2$  we have:

$$\frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} = 0 \quad (100)$$

as expected. Including all the higher order terms,  $F_2, F_3, \dots$  etc, we would obtain a set of differential equations. Solving them we would be able to write the complete series (91). In practice, since  $\sigma$  is small, this series is dominated by the first terms and it is often sufficient to compute only  $F_1$ .

From this simple exercise we conclude that for a nonlinear wave equation such as (94) it is necessary to introduce the “stretching” operators (96):

$$\frac{\partial}{\partial t} = \sigma^{3/2} \frac{\partial}{\partial \tau} \quad \text{and also} \quad \frac{\partial}{\partial x} = \sigma^{1/2} \frac{\partial}{\partial \xi} \quad (101)$$

which are related to the “stretched” coordinates:

$$\xi = \sigma^{1/2} x \quad \text{and} \quad \tau = \sigma^{3/2} t \quad (102)$$

If there was a dissipative term in (94), the wave equation would have the following form:

$$\frac{\partial F}{\partial t} + \alpha F \frac{\partial F}{\partial x} = \nu \frac{\partial^2 F}{\partial x^2} \quad (103)$$

which is called Burgers equation with a dissipative (viscous) coefficient  $\nu$ . It becomes, after performing the expansion (91) and keeping only terms up to order  $\sigma^2$ :

$$\sigma \frac{\partial F_1}{\partial t} + \sigma^2 \frac{\partial F_2}{\partial t} + \sigma \alpha F_0 \frac{\partial F_1}{\partial x} + \sigma^2 \alpha F_0 \frac{\partial F_2}{\partial x} + \sigma^2 \alpha F_1 \frac{\partial F_1}{\partial x} = \nu \sigma \frac{\partial^2 F_1}{\partial x^2} + \nu \sigma^2 \frac{\partial^2 F_2}{\partial x^2} \quad (104)$$

As before, we try to find in equation (104) a structure similar to (103). Keeping the previous choice  $n = 3/2$ ,  $m = 1/2$  and applying (101) to (104) we find:

$$\sigma^{5/2} \left[ \frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} \right] + \sigma^{7/2} \left[ \frac{\partial F_2}{\partial \tau} \right] = \nu \sigma^2 \frac{\partial^2 F_1}{\partial \xi^2} + \nu \sigma^3 \frac{\partial^2 F_2}{\partial \xi^2} \quad (105)$$

When dissipative effects are included we perform the following transformation in the dissipation coefficient [44, 45]:

$$\nu = \sigma^{1/2} \tilde{\nu} \quad (106)$$

in (105), resulting in:

$$\sigma^{5/2} \left[ \frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} \right] + \sigma^{7/2} \left[ \frac{\partial F_2}{\partial \tau} \right] = \tilde{\nu} \sigma^{5/2} \frac{\partial^2 F_1}{\partial \xi^2} + \tilde{\nu} \sigma^{7/2} \frac{\partial^2 F_2}{\partial \xi^2} \quad (107)$$

As before, we divide the last equation by  $\sigma^{1/2}$ , obtaining:

$$\sigma^2 \left[ \frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} \right] + \sigma^3 \left[ \frac{\partial F_2}{\partial \tau} \right] = \tilde{\nu} \sigma^2 \frac{\partial^2 F_1}{\partial \xi^2} + \tilde{\nu} \sigma^3 \frac{\partial^2 F_2}{\partial \xi^2} \quad (108)$$

The terms proportional to  $\sigma^3$  in (108) are neglected and from the  $\sigma^2$  terms we have:

$$\frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} = \tilde{\nu} \frac{\partial^2 F_1}{\partial \xi^2} \quad (109)$$

which is a Burgers equation like (103). In addition, we conclude that when a dissipative term is present in a nonlinear wave equation, apart from (101) and (102), we also need the transformation (106) for the dissipation coefficient.

If a more complete wave equation is considered, i.e., when nonlinear, dissipative and dispersive terms are present, we have the Korteweg-de Vries Burgers (KdV-B) equation:

$$\frac{\partial F}{\partial t} + \alpha F \frac{\partial F}{\partial x} + \beta \frac{\partial^3 F}{\partial x^3} = \nu \frac{\partial^2 F}{\partial x^2} \quad (110)$$

Now,  $\beta$  is the dispersive coefficient. Performing the same calculation we naturally find:

$$\sigma^2 \left[ \frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} + \beta \frac{\partial^3 F_1}{\partial \xi^3} \right] = \sigma^2 \left[ \tilde{\nu} \frac{\partial^2 F_1}{\partial \xi^2} \right] \quad (111)$$

which gives the KdV-B:

$$\frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} + \beta \frac{\partial^3 F_1}{\partial \xi^3} = \tilde{\nu} \frac{\partial^2 F_1}{\partial \xi^2} \quad (112)$$

When more dimensions and different coordinate systems are considered, the procedure can be systematically improved.

### C. Some special cases

In the framework of the RPM we transport the equations of hydrodynamics from the space of cartesian, spherical or cylindrical coordinates to the space of the “stretched coordinates”. Some well known equations and the coordinate transformations and expansions required to obtain them are given below.

#### 1. One dimensional KdV equation

We write this equation in cartesian coordinates ( $x$ ), in radial cylindrical coordinate and in radial spherical coordinate ( $r$ ). The corresponding “stretched coordinates” are given by  $\xi$  for space and  $\tau$  for time as: [20, 35–40, 46]:

$$\xi = \frac{\sigma^{1/2}}{L}(\mathcal{X} - c_s t) \quad (113)$$

and

$$\tau = \frac{\sigma^{3/2}}{L} c_s t \quad (114)$$

where  $\mathcal{X} = x$  or  $\mathcal{X} = r$ . The above transformation of coordinates must be made simultaneously with the following expansions of the baryon density, energy density and fluid velocity around their equilibrium values:

$$\hat{\rho} = \frac{\rho_B}{\rho_0} = 1 + \sigma \rho_1 + \sigma^2 \rho_2 + \dots \quad (115)$$

$$\hat{\varepsilon} = \frac{\varepsilon}{\varepsilon_0} = 1 + \sigma \varepsilon_1 + \sigma^2 \varepsilon_2 + \dots \quad (116)$$

$$\hat{v} = \frac{v}{c_s} = \sigma v_1 + \sigma^2 v_2 + \dots \quad (117)$$

For two and three dimensional nonlinear wave equations we will use the “stretched coordinates” presented in [42, 43] and references therein.

## 2. Two dimensional cylindrical breaking wave equation

In this case the coordinate transformation is given by:

$$R = \frac{\sigma^{1/2}}{L}(r - c_s t) \ , \quad Z = \frac{\sigma}{L}z \ , \quad T = \frac{\sigma^{3/2}}{L}c_s t \quad (118)$$

The energy density and fluid velocity components are expanded around their equilibrium values:

$$\hat{\varepsilon} = \frac{\varepsilon}{\varepsilon_0} = 1 + \sigma \varepsilon_1 + \sigma^2 \varepsilon_2 + \sigma^3 \varepsilon_3 + \dots \quad (119)$$

$$\hat{v}_r = \frac{v_r}{c_s} = \sigma v_{r1} + \sigma^2 v_{r2} + \sigma^3 v_{r3} + \dots \quad (120)$$

$$\hat{v}_z = \frac{v_z}{c_s} = \sigma^{3/2} v_{z1} + \sigma^{5/2} v_{z2} + \sigma^{7/2} v_{z3} + \dots \quad (121)$$

## 3. Three dimensional cylindrical KP equation

In this case we have:

$$R = \frac{\sigma^{1/2}}{L}(r - c_s t) \ , \quad \Phi = \sigma^{-1/2} \varphi \ , \quad Z = \frac{\sigma}{L}z \ , \quad T = \frac{\sigma^{3/2}}{L}c_s t \quad (122)$$

and for the baryonic density and fluid velocity components expansions are:

$$\hat{\rho} = \frac{\rho_B}{\rho_0} = 1 + \sigma \rho_1 + \sigma^2 \rho_2 + \sigma^3 \rho_3 + \dots \quad (123)$$

$$\hat{v}_r = \frac{v_r}{c_s} = \sigma v_{r1} + \sigma^2 v_{r2} + \sigma^3 v_{r3} + \dots \quad (124)$$

$$\hat{v}_\varphi = \frac{v_\varphi}{c_s} = \sigma^{3/2} v_{\varphi1} + \sigma^{5/2} v_{\varphi2} + \sigma^{7/2} v_{\varphi3} + \dots \quad (125)$$

$$\hat{v}_z = \frac{v_z}{c_s} = \sigma^{3/2} v_{z1} + \sigma^{5/2} v_{z2} + \sigma^{7/2} v_{z3} + \dots \quad (126)$$

$$\hat{\rho}^{4/3} = [1 + (\sigma \rho_1 + \sigma^2 \rho_2 + \dots)]^{4/3} \cong 1 + \frac{4}{3} \sigma \rho_1 + \frac{4}{3} \sigma^2 \rho_2 + \dots \quad (127)$$

$$\hat{\rho}^{1/3} = [1 + (\sigma \rho_1 + \sigma^2 \rho_2 + \dots)]^{1/3} \cong 1 + \frac{1}{3} \sigma \rho_1 + \frac{1}{3} \sigma^2 \rho_2 + \dots \quad (128)$$

#### 4. Three dimensional cartesian KP equation

Here the stretched coordinates are:

$$X = \frac{\sigma^{1/2}}{L}(x - c_s t) , \quad Y = \frac{\sigma}{L}y , \quad Z = \frac{\sigma}{L}z , \quad T = \frac{\sigma^{3/2}}{L}c_s t \quad (129)$$

and the expansions are:

$$\hat{\rho} = \frac{\rho_B}{\rho_0} = 1 + \sigma \rho_1 + \sigma^2 \rho_2 + \sigma^3 \rho_3 + \dots \quad (130)$$

$$\hat{v}_x = \frac{v_x}{c_s} = \sigma v_{x1} + \sigma^2 v_{x2} + \sigma^3 v_{x3} + \dots \quad (131)$$

$$\hat{v}_y = \frac{v_y}{c_s} = \sigma^{3/2} v_{y1} + \sigma^2 v_{y2} + \sigma^{5/2} v_{y3} + \dots \quad (132)$$

$$\hat{v}_z = \frac{v_z}{c_s} = \sigma^{3/2} v_{z1} + \sigma^2 v_{z2} + \sigma^{5/2} v_{z3} + \dots \quad (133)$$

$$\hat{\rho}^{4/3} \cong 1 + \frac{4}{3}\sigma \rho_1 + \frac{4}{3}\sigma^2 \rho_2 + \dots \quad (134)$$

$$\hat{\rho}^{1/3} \cong 1 + \frac{1}{3}\sigma \rho_1 + \frac{1}{3}\sigma^2 \rho_2 + \dots \quad (135)$$

In all cases  $L$  is a characteristic length scale of the problem,  $\varepsilon_0$  is the equilibrium (or reference) energy density,  $\rho_0$  is the equilibrium (or reference) baryon density and  $c_s$  is the speed of sound. When viscosity is included, we perform the transformation (106) as in [44, 45]:

$$\zeta = \sigma^{1/2} \tilde{\zeta} \quad \text{and also} \quad \eta = \sigma^{1/2} \tilde{\eta} \quad (136)$$

Once the equations are in the “stretched spaces” and expanded, we neglect terms proportional to  $\sigma^n$  for  $n > 2$  and organize the equations as series in powers of  $\sigma$ ,  $\sigma^{3/2}$  and  $\sigma^2$ . These equations form a system of differential equations which are combined to yield the final nonlinear equation for the relevant perturbation. Finally we transform the nonlinear equation back to the cartesian (cylindrical or spherical) space and solve it.

## IV. THE EQUATION OF STATE

The equations of hydrodynamics discussed in the previous sections must be supplemented with an equation of state (EOS), i.e., a relation between pressure ( $p$ ) and energy density ( $\varepsilon$ ) or matter density ( $\rho$ ). In relativistic hydrodynamics, we can write the EOS as:

$$p = c_s^2 \varepsilon \quad (137)$$

where the speed of sound  $c_s$  must be smaller than one. The equation of state is derived from microscopic theories of the strongly interacting system. As mentioned above, the fundamental theory of strong interactions (QCD) predicts that cold and/or dilute systems are in the hadronic phase, where the degrees of freedom are baryons (proton, neutron,  $\Delta$ , ...) and mesons ( $\pi$ ,  $\rho$ , ...). At higher densities and/or temperatures, there is a phase transition and the formation of the quark gluon plasma (QGP), a phase where these particles are free and all hadrons are dissolved. In all phases we may have thermal equilibrium and hydrodynamical behavior. In both hadron and quark-gluon fluids we may have excitations and the nonlinear propagation of perturbations. In what follows we shall study the formation of these nonlinear waves and the role played by the equation of state.

In the pioneering study of Refs. [20, 40] the authors studied the equations of non-relativistic hydrodynamics and, with the help of the RPM, they were able to derive a KdV (Korteweg de-Vries) equation for a perturbation in the nuclear density. A key ingredient in that work was the equation of state, which established the following relation between pressure ( $p$ ) and density ( $\rho$ ):

$$\frac{\vec{\nabla} p}{\rho} = \vec{\nabla} \phi \quad (138)$$

where  $\phi$  is a potential, playing the role of heat function, given by:

$$\phi = \frac{1}{\rho_0} [c_1 \rho' + c_2 \vec{\nabla}^2 \rho' + \dots] \quad (139)$$

where  $c_1$  and  $c_2$  are constants and  $\rho'$  is the deviation of the density from its equilibrium value  $\rho' = \rho - \rho_0$ . The Laplacian in (139), combined with the gradient in (138) gives origin to the cubic derivative of the KdV equation. As a result the authors arrived at the conclusion that a KdV soliton may exist in cold nuclear matter, when, for example, a light nucleus is impinged on a heavy nucleus. This conclusion relied strongly on (139) and (138), which come from an oversimplified description of the nuclear interactions. In [35–38] the nuclear interactions were described in terms of a relativistic mean field model, which is a variant of the Walecka model. Within this framework we may try to answer the question: what is the microscopic origin of higher order derivative terms appearing in the equation of state? This will be discussed in the following sections.



## A. Hadronic Matter

In this subsection we present an equation of state derived from a successful relativistic mean field model: Quantum Hadrodynamics (QHD) or Walecka Model. For a modern approach, using chiral power counting in an effective field theory for nuclear matter with nucleons and pions as degrees of freedom, see Ref. [47].

This model is well established and is the subject of textbooks as Refs. [3]. The Lagrangian density of nonlinear QHD is given by:

$$\begin{aligned} \mathcal{L} = & \bar{\psi}[\gamma_\mu(i\partial^\mu - g_V V^\mu) - (M - g_S\phi)]\psi + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_S^2\phi^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \\ & + \frac{1}{2}m_V^2V_\mu V^\mu - \frac{b}{3}\phi^3 - \frac{c}{4}\phi^4 + \mathcal{L}_d \end{aligned} \quad (140)$$

where  $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$  and:

$$\mathcal{L}_d = d \frac{g_V}{m_V^2} \bar{\psi}(\partial_\nu \partial^\nu V_\mu) \gamma^\mu \psi \quad (141)$$

Except for the last term, this is the standard nonlinear QHD Lagrangian which is able to reproduce all the main features of nuclear matter and finite nuclei. This last term was added only in [35–38] and illustrates how to include higher order derivative terms in the equation of state.

The Lagrangian (141) is the modern version of (139) and (138). In (140) the degrees of freedom are the baryon field  $\psi$ , the neutral scalar meson field  $\phi$  and the neutral vector meson field  $V_\mu$ , with the respective couplings and masses. The last and new term (141) is designed to be small in comparison with the main baryon-vector meson interaction term  $g_V \bar{\psi} \gamma_\mu V^\mu \psi$ . Because of the derivatives, it is of the order of:

$$\frac{p^2}{m_V^2} \sim \frac{k_F^2}{m_V^2} \sim 0.12 \quad (142)$$

where the Fermi momentum is  $k_F \simeq 0.28$  GeV and  $m_V \simeq 0.8$  GeV. The form chosen for the new interaction term is not dictated by any symmetry argument, has no other deep justification and is just one possible interaction term among many others. It is used here as a prototype to study the effects of higher derivative terms, which, as it will be seen, may generate more complex wave equations, such as KdV. The parameter  $d$  is free and plays the role of a “marker”. Setting  $d$  equal to zero switches off the new term and we recover the usual QHD. On the other hand  $d = 1$  means that the coupling  $g_V$  is the standard one. Other

values imply a correction in this coupling. As mentioned in the beginning of this section, interesting phenomena, such as KdV solitons, may appear as a consequence of the use of equations of state with higher order derivative terms.

Baryon number propagation in nuclear matter satisfies the diffusion equation:

$$\frac{\partial \rho_B}{\partial t} = D \nabla^2 \rho_B \quad (143)$$

where the diffusion constant  $D$  has been numerically calculated and studied as a function of density and temperature. For example, in [48] it was found that  $D \simeq 0.35 \text{ fm}$  at densities comparable to the equilibrium nuclear density and temperatures of the order of  $80 \text{ MeV}$ . This number is small compared to any nuclear size scale and can be interpreted as indicating that

$$\frac{\partial \rho_B}{\partial t} \ll \nabla^2 \rho_B \quad (144)$$

and therefore the density gradients do not disappear very rapidly in nuclear matter. Because of the above inequality we can neglect the time derivatives in (141).

The usual mean field theory (MFT) approximation is based on two assumptions: a) the baryonic sources are intense and their coupling to the meson fields is strong and b) infinite nuclear matter is static, homogeneous in space and isotropic.

The first assumption above is implemented performing the substitutions:

$$V_\mu \rightarrow \langle V_\mu \rangle \equiv \delta_{\mu 0} V_0 \quad (145)$$

and

$$\phi \rightarrow \langle \phi \rangle \equiv \phi_0 \quad (146)$$

in (140) and obtaining  $\mathcal{L}^* = \mathcal{L}(V_\mu \rightarrow \langle V_\mu \rangle \equiv \delta_{\mu 0} V_0 \quad \phi \rightarrow \langle \phi \rangle \equiv \phi_0)$ :

$$\begin{aligned} \mathcal{L}^* = & \bar{\psi}[(i\gamma_\mu \partial^\mu - g_V \gamma_0 V_0) - (M - g_S \phi_0)]\psi + \frac{1}{2}(\partial_\mu \phi_0 \partial^\mu \phi_0 - m_S^2 \phi_0^2) + \\ & + \frac{1}{2}(\vec{\nabla} V_0)^2 + \frac{1}{2}m_V^2 V_0^2 - \frac{b}{3}\phi_0^3 - \frac{c}{4}\phi_0^4 \end{aligned} \quad (147)$$

The equations of motion are given by [38, 49–51]:

$$\frac{\partial \mathcal{L}^*}{\partial \eta_i} - \partial_\mu \frac{\partial \mathcal{L}^*}{\partial (\partial_\mu \eta_i)} + \partial_\nu \partial_\mu \left[ \frac{\partial \mathcal{L}^*}{\partial (\partial_\mu \partial_\nu \eta_i)} \right] = 0 \quad (148)$$

where  $\eta_i = \Psi, \phi_0, V_0$  and read:

$$-\vec{\nabla}^2 V_0 + m_V^2 V_0 = g_V \bar{\psi} \gamma^0 \psi \quad (149)$$

$$(\partial_\mu \partial^\mu + m_S^2)\phi_0 = g_S \bar{\psi}\psi - b\phi_0^2 - c\phi_0^3 \quad (150)$$

$$\left[ i\gamma_\mu \partial^\mu - g_V \gamma_0 V_0 - (M - g_S \phi_0) \right] \Psi = 0 \quad (151)$$

The effective mass of the nucleon is given by  $M^* = M - g_S \phi_0$ . In the case of pure QHD (147), because of the interaction between the nucleon and the vector meson, we can anticipate that the second order derivative term in the field  $V^\mu$ , which comes from its equation of motion, may be transferred to the term  $\bar{\psi}\gamma_\mu\psi$  and hence to the baryon density  $\rho_B$ . In the mean field approximation, only the time component of the field,  $V^0$ , contributes, yielding the term  $\vec{\nabla}^2 V_0$ . It is possible to estimate  $\vec{\nabla}^2 V_0$  from the equation of motion (149). To do so, we first rewrite the equation (149) as function of the baryon density:

$$-\vec{\nabla}^2 V_0 + m_V^2 V_0 = g_V \rho_B \quad (152)$$

Next we assume that  $\vec{\nabla}^2 V_0/m_V^2 \ll V_0$ , neglecting the Laplacian in (152) to find the first order estimate of  $V_0$ :

$$V_0 = \frac{g_V}{m_V^2} \rho_B \quad (153)$$

An improvement of this estimate of  $V_0$  is obtained taking the Laplacian of (153) and substituting it in the Laplacian present in (152). After this simple algebraic procedure, we solve the resulting equation for  $V_0$  and obtain:

$$V_0 = \frac{g_V}{m_V^2} \rho_B + \frac{g_V}{m_V^4} \vec{\nabla}^2 \rho_B \quad (154)$$

The energy-momentum tensor is given by [38, 49–51]:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \eta_i)} (\partial^\nu \eta_i) - g^{\mu\nu} \mathcal{L}^* - \left[ \partial_\beta \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \partial_\beta \eta_i)} \right] (\partial^\nu \eta_i) + \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \partial_\beta \eta_i)} (\partial_\beta \partial^\nu \eta_i) \quad (155)$$

The energy density is:

$$\varepsilon = \langle T_{00} \rangle \quad (156)$$

which becomes [38] :

$$\varepsilon = \frac{1}{2}(\partial_0 \phi_0)^2 + \frac{1}{2}(\vec{\nabla} \phi_0)^2 + \frac{1}{2}g_V \rho_B V_0 + \frac{1}{2}m_S^2 \phi_0^2 + \frac{b}{3}\phi_0^3 + \frac{c}{4}\phi_0^4 + \frac{\gamma_s}{(2\pi)^3} \int_0^{k_F} d^3k (\vec{k}^2 + M^{*2})^{1/2} \quad (157)$$

Inserting (154) in the expression of the energy density (157) and also writing  $\phi_0$  in terms of  $M^*$  we find the following expression:

$$\varepsilon = \frac{g_V^2}{2m_V^2} \rho_B^2 + \frac{g_V^2}{2m_V^4} \rho_B \vec{\nabla}^2 \rho_B + \frac{m_S^2}{2g_S^2} (M - M^*)^2$$

$$+ b \frac{(M - M^*)^3}{3g_S^3} + c \frac{(M - M^*)^4}{4g_S^4} + \frac{\gamma_s}{(2\pi)^3} \int_0^{k_F} d^3k (\vec{k}^2 + M^{*2})^{1/2} \quad (158)$$

In a first approximation, the variables  $\rho_B$ ,  $\vec{\nabla}^2 \rho_B$  and  $M^*$  are independent from each other and therefore, taking the derivative of the above expression with respect to  $\rho_B$  we have:

$$\frac{\partial \varepsilon}{\partial \rho_B} = \frac{g_V^2}{m_V^2} \rho_B + \frac{g_V^2}{2m_V^4} \vec{\nabla}^2 \rho_B \quad (159)$$

We will add the two sources of inhomogeneities in  $\rho_B$  (which are responsible for a non-vanishing  $\vec{\nabla}^2 \rho_B$ ). For cold nuclear matter we have then [38]:

$$\begin{aligned} \varepsilon = & \frac{1}{2} \left\{ \frac{\partial}{\partial t} \left[ \frac{(M - M^*)}{g_S} \right] \right\}^2 + \frac{1}{2} \left\{ \vec{\nabla} \left[ \frac{(M - M^*)}{g_S} \right] \right\}^2 + \frac{m_S^2}{2g_S^2} (M - M^*)^2 + \\ & b \frac{(M - M^*)^3}{3g_S^3} + c \frac{(M - M^*)^4}{4g_S^4} + \frac{g_V^2}{2m_V^2} \rho_B^2 + \left( d + \frac{1}{2} \right) \frac{g_V^2}{m_V^4} \rho_B \vec{\nabla}^2 \rho_B + \\ & + \frac{\gamma_s}{(2\pi)^3} \int_0^{k_F} d^3k (\vec{k}^2 + M^{*2})^{1/2} \end{aligned} \quad (160)$$

where  $\gamma_s = 4$  is the nucleon degeneracy factor. For hot nuclear matter, the energy density as described in [38] is given by:

$$\begin{aligned} \varepsilon = & \frac{1}{2} \left\{ \frac{\partial}{\partial t} \left[ \frac{(M - M^*)}{g_S} \right] \right\}^2 + \frac{1}{2} \left\{ \vec{\nabla} \left[ \frac{(M - M^*)}{g_S} \right] \right\}^2 + \frac{m_S^2}{2g_S^2} (M - M^*)^2 + \\ & b \frac{(M - M^*)^3}{3g_S^3} + c \frac{(M - M^*)^4}{4g_S^4} + \frac{g_V^2}{2m_V^2} \rho_B^2 + \left( d + \frac{1}{2} \right) \frac{g_V^2}{m_V^4} \rho_B \vec{\nabla}^2 \rho_B + \\ & + \frac{\gamma_s}{(2\pi)^3} \int d^3k \ h_+ [n_{\vec{k}}(T, \nu) + \bar{n}_{\vec{k}}(T, \nu)] \end{aligned} \quad (161)$$

The baryon density is:

$$\rho_B = \frac{\gamma_s}{(2\pi)^3} \int d^3k [n_{\vec{k}}(T, \nu) - \bar{n}_{\vec{k}}(T, \nu)] \quad (162)$$

with

$$n_{\vec{k}}(T, \nu) \equiv \frac{1}{1 + e^{(h_+ - \nu)/T}} \quad (163)$$

$$\bar{n}_{\vec{k}}(T, \nu) \equiv \frac{1}{1 + e^{(h_+ + \nu)/T}} \quad (164)$$

$$\nu \equiv \mu_B - g_V V_0 + d \frac{g_V}{m_V^2} (\partial^\mu \partial_\nu V_0) \quad (165)$$

and

$$h_+ \equiv (\vec{k}^2 + M^{*2})^{1/2} \quad (166)$$

It is important to note that the term  $\mathcal{L}_d$  provides the “ $(d+1/2)$ -term” in the energy densities (161) and (160) that generates the KdV equation. The nucleon effective mass ( $M^*$ ) at zero temperature is obtained through the minimization of  $\varepsilon$  with respect to  $M^*$ :

$$\frac{\partial \varepsilon}{\partial M^*} = 0$$

which, with the help of (160) yields:

$$\begin{aligned} M^* = M - \frac{g_S^2}{m_S^2} \frac{\gamma_s}{(2\pi)^3} \int_0^{k_F} d^3k \frac{M^*}{(\vec{k}^2 + M^{*2})^{1/2}} \\ + \frac{g_S^2}{m_S^2} \left[ \frac{b}{g_S^3} (M - M^*)^2 + \frac{c}{g_S^4} (M - M^*)^3 \right] \end{aligned} \quad (167)$$

Analogously, at finite temperature:

$$\begin{aligned} M^* = M - \frac{g_S^2}{m_S^2} \frac{\gamma_s}{(2\pi)^3} \int d^3k \frac{M^*}{h_+} \left[ n_{\vec{k}}(T, \nu) + \bar{n}_{\vec{k}}(T, \nu) \right] \\ + \frac{g_S^2}{m_S^2} \left[ \frac{b}{g_S^3} (M - M^*)^2 + \frac{c}{g_S^4} (M - M^*)^3 \right] \end{aligned} \quad (168)$$

and we conclude by these two last expressions that “ $d$ -term” does not affect the nucleon effective mass.

## B. Quark Gluon Plasma

The idea that quarks and gluons may exist as free particles in a deconfined phase was advanced long ago [52] in the context of compact stars. In these objects gravitation is strong enough to compress the nucleons and make them overlap with each other. The distance between the quarks can be so small that, due to asymptotic freedom, they almost do not interact. In this picture, we may have large regions of space populated with free quarks. This state is called the (cold) quark gluon plasma. This naive description of QCD in extreme conditions of density was later revisited with much more sophisticated models. Quarks and gluons may also form a hot quark gluon plasma. The hot QGP has been extensively studied in lattice simulations and also in heavy ion experiments at CERN and LHC. Today we know that a hot and deconfined phase is formed in the existing accelerators, but it is far more complicated than previously imagined. In particular the quarks and gluons are not really free. Instead, they still interact strongly with each other forming a state called the strongly

interacting QGP, or sQGP. Also the non-trivial vacuum structure persists until relatively large temperatures.

A starting point to study in a unified way both the cold and hot QGP is the MIT bag model [53]. According to this model in its simplest version, massless and non-interacting quarks live in a spherical cavity (the “bag”) in the physical QCD vacuum, which is a medium. The confining property is represented by a constant term, “B”, called the bag constant. In the next subsection we shall briefly show how to calculate the QGP equation of state with the help of this model.

### 1. The MIT Bag Model

In what follows we consider only quarks  $u$  and  $d$ . Each quark has three color states and they are massless. We also have eight massless gluons and we neglect the interactions in the QGP. The confinement property is included in the model through the introduction of a constant, positive energy per unit volume in the vacuum:

$$\mathcal{B} = \left( \frac{E}{V} \right)_{vac} \quad (169)$$

that can be interpreted as the energy needed to create a bubble or bag in the vacuum, in which the noninteracting quarks and gluons are confined.  $\mathcal{B}$  is known as “bag constant”.

The parameter  $\mathcal{B}$  can be extracted from a phenomenological analysis of hadron spectroscopy or from lattice QCD calculations. There is a relationship between  $\mathcal{B}$  and the critical temperature of quark-hadron transition  $T_c$  which is determined by considering that during the phase transition the pressure is zero.

The quarks have baryon number  $1/3$  and the chemical potential for gluons is zero. The baryon density, energy density and pressure for the QGP are given by:

$$\rho_B = \frac{1}{3} \frac{\gamma_Q}{(2\pi)^3} \int d^3k [n_{\vec{k}} - \bar{n}_{\vec{k}}] \quad (170)$$

where

$$n_{\vec{k}} \equiv n_{\vec{k}}(T) = \frac{1}{1 + e^{(k - \frac{1}{3}\mu_B)/T}} \quad (171)$$

and

$$\bar{n}_{\vec{k}} \equiv \bar{n}_{\vec{k}}(T) = \frac{1}{1 + e^{(k + \frac{1}{3}\mu_B)/T}} \quad (172)$$

where  $\mu_B$  is the baryon chemical potential. The energy density is given by:

$$\varepsilon = \mathcal{B} + \frac{\gamma_G}{(2\pi)^3} \int d^3k \, k \, (e^{k/T} - 1)^{-1} + \frac{\gamma_Q}{(2\pi)^3} \int d^3k \, k \, [n_{\vec{k}} + \bar{n}_{\vec{k}}] \quad (173)$$

where the first term of the expression above is the gluon contribution. The pressure is given by:

$$p = -\mathcal{B} + \frac{1}{3} \left\{ \frac{\gamma_G}{(2\pi)^3} \int d^3k \, k \, (e^{k/T} - 1)^{-1} + \frac{\gamma_Q}{(2\pi)^3} \int d^3k \, k \, [n_{\vec{k}} + \bar{n}_{\vec{k}}] \right\} \quad (174)$$

The degeneracy factors are:

$$\gamma_G = 2(\text{polarizations}) \times 8(\text{colors}) = 16 \quad \text{for gluons} \quad (175)$$

and

$$\gamma_Q = 2(\text{spins}) \times 2(\text{flavours}) \times 3(\text{colors}) = 12 \quad \text{for quarks} \quad (176)$$

The integral of the gluon distribution function can be calculated analytically and the thermodynamics of QGP can be summarized in the following expressions:

$$\rho_B = \frac{2}{\pi^2} \int_0^\infty dk \, k^2 [n_{\vec{k}} - \bar{n}_{\vec{k}}] \quad (177)$$

and

$$3(p + \mathcal{B}) = \varepsilon - \mathcal{B} = \frac{8\pi^2}{15} T^4 + \frac{6}{\pi^2} \int_0^\infty dk \, k^3 [n_{\vec{k}} + \bar{n}_{\vec{k}}] \quad (178)$$

which provide us with the EOS of QGP with all baryon densities and at all temperatures:

$$p = \frac{1}{3}\varepsilon - \frac{4}{3}\mathcal{B} \quad (179)$$

The sound speed  $c_s$  is given by:

$$c_s^2 = \frac{\partial p}{\partial \varepsilon} = \frac{1}{3} \quad (180)$$

## 2. The cold QGP

In the core of a neutron star, the temperature is zero and baryon density is considerable. This is a good place to study baryon density perturbations. The quark distribution function becomes the step function with  $\mu_B = 3k_F$  and (177) becomes:

$$\rho_B = \frac{\gamma_s}{6\pi^2} k_F^3 \quad (181)$$

Expression (178) becomes:

$$3(p + \mathcal{B}) = \varepsilon - \mathcal{B} = \frac{6}{\pi^2} \left( \frac{k^4}{4} \Big|_0^{k_F} \right) = \frac{3}{2\pi^2} k_F^4 \quad (182)$$

Using (181) we find:

$$3(p + \mathcal{B}) = \varepsilon - \mathcal{B} = \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{4/3}$$

and so

$$\varepsilon(\rho_B) = \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{4/3} + \mathcal{B} \quad (183)$$

Inserting (183) into (179) we find:

$$p(\rho_B) = \frac{1}{3} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{4/3} - \mathcal{B} \quad (184)$$

From (183) and (184) is possible to rewrite the sum of energy density and pressure as:

$$\varepsilon + p = \frac{4}{3} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{4/3} \quad (185)$$

From (179) we have:

$$\vec{\nabla} p = \frac{1}{3} \vec{\nabla} \varepsilon \quad \text{and also} \quad \frac{\partial p}{\partial t} = \frac{1}{3} \frac{\partial \varepsilon}{\partial t} \quad (186)$$

Using (183) in (186) we find:

$$\vec{\nabla} p = \frac{4}{9} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{1/3} \vec{\nabla} \rho_B \quad (187)$$

and

$$\frac{\partial p}{\partial t} = \frac{4}{9} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{1/3} \frac{\partial \rho_B}{\partial t} \quad (188)$$

### 3. The hot QGP

The hot QGP is formed in heavy ion collisions in the central rapidity region where we have  $\rho_B = 0$ . For our purposes the most relevant physical quantity is the energy density  $\varepsilon$ . Since  $\rho_B = 0$ , the baryon chemical potential is zero ( $\mu_B = 0$ ) and so the distribution functions given by (171) and (172) are the same:

$$n_{\vec{k}} = \bar{n}_{\vec{k}} = \frac{1}{1 + e^{k/T}}$$

and then (178) takes the form:

$$3(p + \mathcal{B}) = \varepsilon - \mathcal{B} = \frac{8\pi^2}{15} T^4 + \frac{12}{\pi^2} \int_0^\infty dk \frac{k^3}{[1 + e^{k/T}]} \quad (189)$$



We can perform the above integral analytically to find:

$$3(p + \mathcal{B}) = \varepsilon - \mathcal{B} = \frac{37}{30}\pi^2 T^4 \quad (190)$$

From basic thermodynamics we know that

$$s = \left( \frac{\partial p}{\partial T} \right)_V \quad (191)$$

which, with the use of (190) leads to the specific form for  $s$ :

$$s = \frac{\partial}{\partial T} \left( -\mathcal{B} + \frac{37}{90}\pi^2 T^4 \right) = 4\frac{37}{90}\pi^2 T^3 \quad (192)$$

The parameter  $\mathcal{B}$ , the “bag constant” can be defined at the temperature  $T_B$ . In the bag surface, (190) is given by:

$$\mathcal{B} = \frac{37}{90}\pi^2 (T_B)^4 \quad (193)$$

Choosing  $\mathcal{B}^{1/4} = 170 \text{ MeV}$  corresponds to  $T_B = 91 \text{ MeV}$ . Rewriting (190) as:

$$T = \left[ \frac{30}{37\pi^2} (\varepsilon - \mathcal{B}) \right]^{1/4} \quad (194)$$

and inserting it into (192) we find:

$$s = s(\varepsilon) = 4\frac{37}{90}\pi^2 \left[ \frac{30}{37\pi^2} (\varepsilon - \mathcal{B}) \right]^{3/4}$$

In a compact notation:

$$s(\varepsilon) = A(\varepsilon - \mathcal{B})^{3/4} \quad (195)$$

where

$$A \equiv 4\frac{37}{90}\pi^2 \left[ \frac{30}{37\pi^2} \right]^{3/4} \quad (196)$$

### C. Mean field theory for Quantum Chromodynamics

In spite of its phenomenological success the MIT bag model gives a poor representation of the quark gluon plasma. From heavy ion collisions there is convincing evidence that quarks and gluons interact strongly forming rather a “perfect fluid” than an ideal gas. Therefore, the picture of free partons must be modified. There are several ways to do that. Here we discuss the approach proposed in [41], which can be called mean field QCD (MFQCD). It allows us to start from QCD Lagrangian and derive an equation of state, which incorporates

the effects of a sizeable strong coupling constant and also residual non-perturbative effects from the QCD vacuum. As an interesting by-product this equation of state supports the existence of KdV solitons in a cold QGP. In what follows we summarize the basic ideas of this approach and derive the expressions for the pressure and energy density, which are relevant for the hydrodynamical study.

We first introduce the mean field approximation for QCD, extending previous works along the same line [54, 55]. We consider a system of quarks and gluons which are represented by the QCD Lagrangian density:

$$\mathcal{L}_{QCD} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \sum_{q=1}^{N_f} \bar{\psi}_i^q \left[ i\gamma^\mu (\delta_{ij}\partial_\mu - igT_{ij}^a G_\mu^a) - \delta_{ij}m_q \right] \psi_j^q \quad (197)$$

with

$$F^{a\mu\nu} = \partial^\mu G^{a\nu} - \partial^\nu G^{a\mu} + gf^{abc}G^{b\mu}G^{c\nu} \quad (198)$$

where  $\psi_i^q$  and  $G_\mu^a$  represent the quark and gluon fields respectively. The summation on  $q$  runs over all quark flavors,  $m_q$  is the mass of the quark of flavor  $q$ ,  $i$  and  $j$  are the color indices of the quarks,  $T^a$  are the SU(3) generators and  $f^{abc}$  are the SU(3) antisymmetric structure constants. For simplicity we consider only light quarks with the same mass  $m$ . Moreover, we drop the summation and consider only one flavor. At the end of our calculation the number of flavors will be recovered. Following [54, 55], we shall start writing the gluon field as:

$$G^{a\mu} = A^{a\mu} + \alpha^{a\mu} \quad (199)$$

where  $A^{a\mu}$  and  $\alpha^{a\mu}$  are the low (“soft”) and high (“hard”) momentum components of the gluon field respectively. We will assume that  $A^{a\mu}$  represents the soft modes which populate the vacuum and  $\alpha^{a\mu}$  represents the modes for which the running coupling constant is small. In a cold quark gluon plasma the density is much larger than the ordinary nuclear matter density. These high densities imply a very large number of sources of the gluon field. Assuming that the coupling constant is not very small, the existence of intense sources implies that the bosonic fields tend to have large occupation numbers at all energy levels, and therefore they can be treated as classical fields. This is the famous approximation for bosonic fields used in relativistic mean field models of nuclear matter [3]. It has been applied to QCD in the past and amounts to assume that the “hard” gluon field,  $\alpha_\mu^a$ , is simply a function of the coordinates [3]:

$$\alpha_\mu^a(\vec{x}, t) = \delta_{\mu 0} \alpha_0^a(\vec{x}, t) \quad (200)$$

with  $\partial_\nu \alpha_\mu^a \neq 0$ . This space and time dependence goes beyond the standard mean field approximation, where  $\alpha_\mu^a$  is constant in space and time and consequently  $\partial_\nu \alpha_\mu^a = 0$ . We keep assuming, as in [41], that the soft gluon field  $A^{a\mu}$  is independent of position and time and thus  $\partial^\nu A^{a\mu} = 0$ . Following the same steps of [41] we obtain the following effective Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\alpha_0^a(\vec{\nabla}^2\alpha_0^a) + \frac{m_G^2}{2}\alpha_0^a\alpha_0^a - \mathcal{B}_{QCD} + \bar{\psi}_i(i\delta_{ij}\gamma^\mu\partial_\mu + g\gamma^0 T_{ij}^a\alpha_0^a - \delta_{ij}m)\psi_j \quad (201)$$

where the constant  $\mathcal{B}_{QCD}$  is the same bag constant for QCD as defined in [41]. In fact, the effective Lagrangian (201) is quite similar to the effective Lagrangian obtained in [41]. The new feature of (201) is the term  $-\frac{1}{2}\alpha_0^a(\vec{\nabla}^2\alpha_0^a)$ . For simplicity, we take the quarks to be massless. From the above Lagrangian, using (155), it is straightforward to derive the energy-momentum tensor  $T_{\mu\nu}$ , which gives us the energy density  $\varepsilon$  and pressure  $p$ :

$$\begin{aligned} \varepsilon = & \left(\frac{27g^2}{16m_G^2}\right)\rho_B^2 + \left(\frac{27g^2}{16m_G^4}\right)\rho_B\vec{\nabla}^2\rho_B + \left(\frac{27g^2}{16m_G^6}\right)\rho_B\vec{\nabla}^2(\vec{\nabla}^2\rho_B) \\ & + \left(\frac{27g^2}{16m_G^8}\right)\vec{\nabla}^2\rho_B\vec{\nabla}^2(\vec{\nabla}^2\rho_B) + \mathcal{B}_{QCD} + 3\frac{\gamma_Q}{2\pi^2}\frac{k_F^4}{4} \end{aligned} \quad (202)$$

and the pressure is:

$$\begin{aligned} p = & \left(\frac{27g^2}{16m_G^2}\right)\rho_B^2 + \left(\frac{9g^2}{4m_G^4}\right)\rho_B\vec{\nabla}^2\rho_B - \left(\frac{9g^2}{8m_G^6}\right)\rho_B\vec{\nabla}^2(\vec{\nabla}^2\rho_B) \\ & - \left(\frac{9g^2}{16m_G^4}\right)\vec{\nabla}\rho_B \cdot \vec{\nabla}\rho_B + \left(\frac{9g^2}{16m_G^6}\right)\vec{\nabla}^2\rho_B\vec{\nabla}^2\rho_B - \left(\frac{9g^2}{8m_G^8}\right)\vec{\nabla}^2\rho_B\vec{\nabla}^2(\vec{\nabla}^2\rho_B) \\ & - \left(\frac{9g^2}{16m_G^8}\right)\vec{\nabla}(\vec{\nabla}^2\rho_B) \cdot \vec{\nabla}(\vec{\nabla}^2\rho_B) - \left(\frac{9g^2}{8m_G^6}\right)\vec{\nabla}\rho_B \cdot \vec{\nabla}(\vec{\nabla}^2\rho_B) \\ & - \mathcal{B}_{QCD} + \frac{\gamma_Q}{2\pi^2}\frac{k_F^4}{4} \end{aligned} \quad (203)$$

where  $\gamma_Q$  is the quark degeneracy factor  $\gamma_Q = 2(\text{spin}) \times 3(\text{flavor}) = 6$  and  $k_F$  is the Fermi momentum defined by the baryon number density by  $\rho_B = k_F^3/\pi^2$ . The other parameters  $g$ ,  $m_G$  and  $\mathcal{B}_{QCD}$  are the coupling of the hard gluons, the dynamical gluon mass and the bag constant in terms of the gluon condensate, respectively. An improved version of the EOS of (202) and (203) was used in the study of three dimensional solitons in cold QGP. These solitons are solutions of the Kadomtsev-Petviashvili (KP) equation, which is the three dimensional generalization of the KdV equation. The complete description of the calculation may be found in [42].

## V. NONLINEAR WAVE EQUATIONS

In the previous sections we presented a review of the equations of hydrodynamics and we introduced the equations of state, which represent the microscopic dynamics of the corresponding fluids. Furthermore, we have presented a mathematical prescription (the RPM) to study perturbations in these fluids preserving the nonlinearities of the differential equations. The application of the RPM to several systems of interest leads to the nonlinear differential equations which we discuss in this section. Most of these wave equations were developed in [35–39, 42, 43, 46].

### A. KdV equation in nuclear matter

#### 1. Cold nuclear matter

We insert (160) in the Euler equation given by (34). We then combine this Euler equation with (57) and follow the RPM procedure to obtain the following KdV equation:

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial \mathcal{X}} + (3 - c_s^2) c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial \mathcal{X}} + \left(d + \frac{1}{2}\right) \left(\frac{g_V^2 \rho_0}{2M m_V^4 c_s}\right) \frac{\partial^3 \hat{\rho}_1}{\partial \mathcal{X}^3} + \mathcal{G} \frac{\hat{\rho}_1}{t} = 0 \quad (204)$$

where  $\mathcal{G}$  is a “geometrical factor”:

$$\mathcal{G} \equiv \begin{cases} 0 & \text{for cartesian coordinates: } \mathcal{X} = x \\ 1 & \text{for spherical coordinates: } \mathcal{X} = r \end{cases} \quad (205)$$

The non-relativistic limit is obtained by the approximation  $3 - c_s^2 \cong 3$  and so (204) becomes:

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial \mathcal{X}} + 3c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial \mathcal{X}} + \left(d + \frac{1}{2}\right) \left(\frac{g_V^2 \rho_0}{2M m_V^4 c_s}\right) \frac{\partial^3 \hat{\rho}_1}{\partial \mathcal{X}^3} + \mathcal{G} \frac{\hat{\rho}_1}{t} = 0 \quad (206)$$

which could have been obtained with the use of the same procedure applied to the nonrelativistic Euler equation (69) with  $\rho = M\rho_B$ .

#### 2. Hot nuclear matter

We follow the same steps listed above, but changing the (160) by the EOS (161) to find a KdV given by:

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial \mathcal{X}} + \left(2 - c_s^2 - \frac{\mu_B m_V^2 c_s^2}{2g_V^2 \rho_0}\right) c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial \mathcal{X}} + \left(d + \frac{1}{2}\right) \left(\frac{c_s}{2m_V^2}\right) \frac{\partial^3 \hat{\rho}_1}{\partial \mathcal{X}^3} + \mathcal{G} \frac{\hat{\rho}_1}{t} = 0 \quad (207)$$

In the last two wave equations  $\hat{\rho}_1 \equiv \sigma\rho_1$  as defined in (115) and  $\mathcal{G}$  is defined by (205). Choosing  $d = -\frac{1}{2}$  in the above equations we eliminate the cubic derivative term and obtain a breaking wave equation.

## B. Breaking wave equation in QGP

### 1. Cold QGP

Inserting the cold QGP MIT EOS, (182) to (188), into the ideal hydrodynamical equations (34) and (57) and following the RPM procedure we obtain the following breaking wave equation:

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \frac{2}{3} c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} = 0 \quad (208)$$

where again  $\hat{\rho}_1 \equiv \sigma\rho_1$ .

### 2. Hot QGP

Now we insert the hot QGP MIT EOS relations, (189) to (196), into the ideal hydrodynamical equations (34) and (55) and following the RPM procedure we obtain the following breaking wave equation:

$$\frac{\partial \hat{\varepsilon}_1}{\partial t} + c_s \frac{\partial \hat{\varepsilon}_1}{\partial x} + \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right] \frac{c_s}{2} \hat{\varepsilon}_1 \frac{\partial \hat{\varepsilon}_1}{\partial x} = 0 \quad (209)$$

where  $\hat{\varepsilon}_1 \equiv \sigma\varepsilon_1$ .

## C. KP and KdV equations from MFQCD

The energy density and pressure are given respectively by (202) and (203) and the wave equations are for perturbations in the baryon density given by  $\hat{\rho}_1 \equiv \sigma\rho_1$  as defined by (123) and (130). We use the ideal hydrodynamical equations (34) and (57) following the RPM procedure. In the calculations we find the following relation:

$$\left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 = \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} = A \quad (210)$$

which relates the speed of sound to the background density  $\rho_0$ :

$$c_s^2 = \frac{\left(\frac{27g^2\rho_0^2}{8m_G^2}\right) + \pi^{2/3}\rho_0^{4/3}}{\left(\frac{27g^2\rho_0^2}{8m_G^2}\right) + 3\pi^{2/3}\rho_0^{4/3}} \quad (211)$$

The wave equations are the following.

### 1. Cylindrical KP equation

$$\begin{aligned} \frac{\partial}{\partial r} \left\{ \frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial r} + \left[ \frac{3}{2}(1 - c_s^2) - \frac{\pi^{2/3}\rho_0^{4/3}}{3A} \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial r} + \left[ \frac{9g^2\rho_0^2 c_s}{8m_G^4 A} \right] \frac{\partial^3 \hat{\rho}_1}{\partial r^3} + \frac{\hat{\rho}_1}{2t} \right\} \\ + \frac{1}{2c_s t^2} \frac{\partial^2 \hat{\rho}_1}{\partial \varphi^2} + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial z^2} = 0 \end{aligned} \quad (212)$$

### 2. KP equation

$$\begin{aligned} \frac{\partial}{\partial x} \left\{ \frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{3}{2}(1 - c_s^2) - \frac{\pi^{2/3}\rho_0^{4/3}}{3A} \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{9g^2\rho_0^2 c_s}{8m_G^4 A} \right] \frac{\partial^3 \hat{\rho}_1}{\partial x^3} \right\} \\ + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial y^2} + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial z^2} = 0 \end{aligned} \quad (213)$$

### 3. KdV equation

The one dimensional cartesian particular case of (213) is obtained by neglecting the  $y$  and  $z$  dependence, so that (213) becomes the KdV :

$$\begin{aligned} \frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{(2 - c_s^2)}{2} - \left( \frac{27g^2\rho_0^2}{8m_G^2} \right) \frac{(2c_s^2 - 1)}{2A} - \frac{\pi^{2/3}\rho_0^{4/3}}{A} \left( c_s^2 - \frac{1}{6} \right) \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} \\ + \left[ \frac{9g^2\rho_0^2 c_s}{8m_G^4 A} \right] \frac{\partial^3 \hat{\rho}_1}{\partial x^3} = 0 \end{aligned} \quad (214)$$

Taking the limit  $m_G \rightarrow \infty$  we obtain from (210) and (211):

$$A = \pi^{2/3}\rho_0^{4/3} \quad , \quad c_s^2 = \frac{1}{3}$$

and (214) becomes:

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \frac{2}{3} c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} = 0 \quad (215)$$

and we recover exactly the result (208), the breaking wave equation for  $\hat{\rho}_1$  at zero temperature in the QGP derived from the MIT equation of state.

#### 4. Breaking wave equation

Neglecting the spatial derivatives in (202) and (203), the equation (214) reduces to:

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{(2 - c_s^2)}{2} - \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) \frac{(2c_s^2 - 1)}{2A} - \frac{\pi^{2/3} \rho_0^{4/3}}{A} \left( c_s^2 - \frac{1}{6} \right) \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} = 0 \quad (216)$$

which is also a breaking wave equation for  $\hat{\rho}_1$  with the  $\rho_0$ ,  $m_G$  and  $g$  dependence in its coefficients.

### D. KP-Burgers equation in hot QGP

The relations (189) to (196) for the hot QGP are now inserted in the relativistic viscous hydrodynamical equations, i.e., the Navier-Stokes (33) and the continuity for the entropy density (53). The ideal case is recovered when the viscous coefficients are set to zero. Following the RPM procedure we obtain the following wave equations:

#### 1. Cylindrical KP-Burgers

$$\begin{aligned} \frac{\partial}{\partial r} \left\{ \frac{\partial \hat{\varepsilon}_1}{\partial t} + c_s \frac{\partial \hat{\varepsilon}_1}{\partial r} + \frac{c_s}{2} \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right] \hat{\varepsilon}_1 \frac{\partial \hat{\varepsilon}_1}{\partial r} + \frac{\hat{\varepsilon}_1}{2t} - \frac{1}{2T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \frac{\partial^2 \hat{\varepsilon}_1}{\partial r^2} \right\} \\ + \frac{1}{2c_s t^2} \frac{\partial^2 \hat{\varepsilon}_1}{\partial \varphi^2} + \frac{c_s}{2} \frac{\partial^2 \hat{\varepsilon}_1}{\partial z^2} = 0 \end{aligned} \quad (217)$$

#### 2. Cylindrical Burgers

Neglecting the  $\varphi$  and  $z$  dependence, the equation (217) becomes:

$$\frac{\partial \hat{\varepsilon}_1}{\partial t} + c_s \frac{\partial \hat{\varepsilon}_1}{\partial r} + \frac{c_s}{2} \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right] \hat{\varepsilon}_1 \frac{\partial \hat{\varepsilon}_1}{\partial r} + \frac{\hat{\varepsilon}_1}{2t} = \frac{1}{2T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \frac{\partial^2 \hat{\varepsilon}_1}{\partial r^2} \quad (218)$$

Setting  $\eta = \zeta = 0$  (ideal fluid) the (218) becomes:

$$\frac{\partial \hat{\varepsilon}_1}{\partial t} + c_s \frac{\partial \hat{\varepsilon}_1}{\partial r} + \frac{c_s}{2} \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right] \hat{\varepsilon}_1 \frac{\partial \hat{\varepsilon}_1}{\partial r} + \frac{\hat{\varepsilon}_1}{2t} = 0 \quad (219)$$

## VI. ANALYTICAL SOLUTIONS OF NONLINEAR WAVE EQUATIONS

We present some cases where particular analytical solutions exist. For the KdV equation we have the soliton solution. Soliton or solitary wave is a localized pulse which propagates without change in shape. For a detailed study of the KdV solitons we recommend the reading of [56]. The soliton will be also used as an initial condition in the study of the numerical solution of the spherical KdV and the breaking wave equations. For the Burgers, cKP, KP and cylindrical KP-Burgers we present the exact solutions as developed in [24, 25, 28, 30, 57–60] where several techniques to solve nonlinear wave equations are presented.

### A. KdV equation in nuclear matter

We first show the soliton solution of the KdV equations (204) and (206) in cold and hot nuclear matter, respectively [35–38]. We have only soliton solutions in the cartesian case  $\mathcal{G} = 0$ , so  $\mathcal{X} = x$ . We also choose  $d = 1/2$  as in [38]. The equation (204) can be integrated and solved exactly and its soliton solution is given by:

$$\hat{\rho}_1(x, t) = \frac{3(u - c_s)}{c_s} (3 - c_s^2)^{-1} \text{sech}^2 \left[ \frac{m_V^2}{g_V} \sqrt{\frac{(u - c_s)c_s M}{2\rho_0}} (x - ut) \right] \quad (220)$$

and the solution of (206) is found by setting  $3 - c_s^2 \cong 3$  in (220):

$$\hat{\rho}_1(x, t) = \frac{(u - c_s)}{c_s} \text{sech}^2 \left[ \frac{m_V^2}{g_V} \sqrt{\frac{(u - c_s)c_s M}{2\rho_0}} (x - ut) \right] \quad (221)$$

Finally, the solution of (207) is:

$$\hat{\rho}_1(x, t) = \frac{3(u - c_s)}{c_s} \left( 2 - c_s^2 - \frac{\mu_B m_V^2 c_s^2}{2g_V^2 \rho_0} \right)^{-1} \text{sech}^2 \left[ \sqrt{\frac{(u - c_s)m_V^2}{2c_s}} (x - ut) \right] \quad (222)$$

### B. KP equations in cold QGP

The cKP equation (212) has the exact analytical soliton solution [42]:

$$\hat{\rho}_1(r, \varphi, z, t) = \frac{h_1}{h_2} \text{sech}^2 \left\{ \frac{\sqrt{h_1}}{2} \left[ ar + bz - \left( u + a \frac{c_s \varphi^2}{2} \right) t \right] \right\} \quad (223)$$

where  $u$  is a parameter which satisfies  $u > ac_s + b^2 c_s / 2a$  and the phase velocity given by  $u + a \frac{c_s \varphi^2}{2}$ . The constants appearing in the above expression are:

$$h_1 = \frac{u - ac_s - b^2 c_s / 2a}{a^3 c_2} \quad \text{and} \quad h_2 = \frac{c_1}{3a^2 c_2} \quad (224)$$



where

$$c_1 \equiv \left[ \frac{3}{2}(1 - c_s^2) - \frac{\pi^{2/3} \rho_0^{4/3}}{3A} \right] c_s \quad (225)$$

and

$$c_2 \equiv \left[ \frac{9g^2 \rho_0^2 c_s}{8m_G^4 A} \right] \quad (226)$$

For the KP equation (213) we have the following soliton solution [42]:

$$\hat{\rho}_1(x, y, z, t) = \frac{3(U - w)}{\mathcal{A}c_1} \text{sech}^2 \left[ \sqrt{\frac{(U - w)}{4\mathcal{A}^3 c_2}} \left( \mathcal{A}x + \mathcal{B}y + \mathcal{C}z - Ut \right) \right] \quad (227)$$

where  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  are real constants and  $w$  is given by:

$$w = \mathcal{A}c_s + \frac{\mathcal{B}^2 c_s}{2} + \frac{\mathcal{C}^2 c_s}{2} \quad (228)$$

We shall consider  $\mathcal{A} > 0$  for simplicity and the parameter  $U$  such that  $U > w$ . For the KdV equation (214) we have:

$$\hat{\rho}_1(x, t) = \frac{3(u - c_s)}{c_3} \text{sech}^2 \left[ \sqrt{\frac{(u - c_s)}{4c_4}} (x - ut) \right] \quad (229)$$

where  $u$  is an arbitrary supersonic velocity and the constants  $c_3$  and  $c_4$  are given by:

$$c_3 \equiv \left[ \frac{(2 - c_s^2)}{2} - \left( \frac{27g^2 \rho_0^2}{m_G^2} \right) \frac{(2c_s^2 - 1)}{2A} - \frac{\pi^{2/3} \rho_0^{4/3}}{A} \left( c_s^2 - \frac{1}{6} \right) \right] c_s \quad (230)$$

and

$$c_4 \equiv \left[ \frac{9g^2 \rho_0^2 c_s}{m_G^4 A} \right] \quad (231)$$

### C. KP-Burgers equation in hot QGP

The cKP-B (217) has the solutions:

$$\begin{aligned} \hat{\varepsilon}_1(r, z, \varphi, t) = & \frac{2DA}{c_s T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right]^{-1} \\ & - \frac{2DA}{c_s T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right]^{-1} \times \\ & \times \tanh \left\{ D \left[ Ar + Bz - A \frac{c_s \varphi^2 t}{2} - \left( Ac_s + \frac{B^2 c_s}{2A} + \frac{DA^2}{T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \right) t \right] \right\} \end{aligned} \quad (232)$$

$$\begin{aligned}
\hat{\varepsilon}_1(r, z, \varphi, t) = & -\frac{2DA}{c_s T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right]^{-1} \\
& - \frac{2DA}{c_s T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right]^{-1} \times \\
& \times \tanh \left\{ D \left[ Ar + Bz - A \frac{c_s \varphi^2 t}{2} - \left( Ac_s + \frac{B^2 c_s}{2A} - \frac{DA^2}{T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \right) t \right] \right\}
\end{aligned} \tag{233}$$

or the following solutions:

$$\begin{aligned}
\hat{\varepsilon}_1(r, z, \varphi, t) = & \frac{2DA}{c_s T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right]^{-1} \\
& - \frac{2DA}{c_s T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right]^{-1} \times \\
& \times \coth \left\{ D \left[ Ar + Bz - A \frac{c_s \varphi^2 t}{2} - \left( Ac_s + \frac{B^2 c_s}{2A} + \frac{DA^2}{T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \right) t \right] \right\}
\end{aligned} \tag{234}$$

$$\begin{aligned}
\hat{\varepsilon}_1(r, z, \varphi, t) = & -\frac{2DA}{c_s T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right]^{-1} \\
& - \frac{2DA}{c_s T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \left[ 1 + \left( \frac{T_B}{T_0} \right)^4 \right]^{-1} \times \\
& \times \coth \left\{ D \left[ Ar + Bz - A \frac{c_s \varphi^2 t}{2} - \left( Ac_s + \frac{B^2 c_s}{2A} - \frac{DA^2}{T_0} \left( \frac{\zeta}{s} + \frac{4\eta}{3s} \right) \right) t \right] \right\}
\end{aligned} \tag{235}$$

where the real constants to be chosen are  $A$ ,  $B$  and  $D$ .

## VII. NUMERICAL SOLUTIONS

In this section we present numerical results. The solutions of the differential equations can be grouped in those which are smooth and those which exhibit some non-smooth behavior, such as rapid oscillations or the formation of “walls”, specially at later times. This kind of behavior appears when there is a lack of balance between the different terms of the equations. Therefore, before presenting numbers and plots, we discuss, in the next subsection, the conditions for finding stable solutions.

### A. Soliton stability

As we have seen, perturbations in fluids with different equations of state generate different nonlinear wave equations. Some examples are the Kadomtsev-Petviashvili (KP) equation:

$$\frac{\partial}{\partial x} \left\{ \frac{\partial F}{\partial t} + \alpha_1 F \frac{\partial F}{\partial x} + \alpha_2 \frac{\partial^3 F}{\partial x^3} \right\} + \alpha_3 \frac{\partial^2 F}{\partial y^2} + \alpha_4 \frac{\partial^2 F}{\partial z^2} = 0. \quad (236)$$

and its particular cases, such as the KdV:

$$\frac{\partial F}{\partial t} + \alpha_1 F \frac{\partial F}{\partial x} + \alpha_2 \frac{\partial^3 F}{\partial x^3} = 0 \quad (237)$$

and the breaking wave equation:

$$\frac{\partial F}{\partial t} + \alpha_1 F \frac{\partial F}{\partial x} = 0 \quad (238)$$

Another example of a nonlinear wave in a dissipative system is the Burgers equation:

$$\frac{\partial F}{\partial t} + \alpha_1 F \frac{\partial F}{\partial x} + \alpha_2 \frac{\partial^2 F}{\partial x^2} = 0 \quad (239)$$

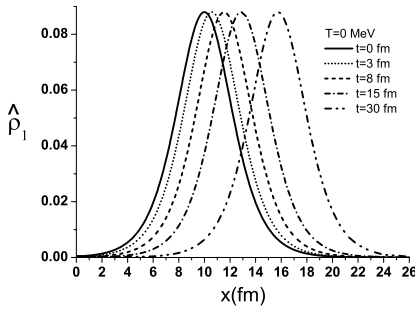
where  $\alpha_1$  to  $\alpha_4$  are real constants. Having derived a particular differential equation, we can check whether the obtained equation is consistent with the physical picture of a small amplitude and long wave length perturbation propagating over large distances. We shall follow the analysis performed in Ref. [18]. Let us assume that the nonlinear equations listed above for a generic function  $F$  have a solitary wave solution with a typical large length  $L \simeq 1/\sigma$  ( $0 < \sigma \ll 1$ ). Considering the general case, the KP equation has a dispersion  $\frac{\partial^4 F}{\partial x^4}$  term that is about  $\frac{\partial^4 F}{\partial x^4} \simeq \sigma^4 F$ . It must arise at a propagation distance (or equivalently propagation time  $T$ )  $D$ , accounted for in the equation by the term  $\frac{\partial^2 F}{\partial x \partial t} \simeq \sigma \frac{F}{T}$ . If both the dispersion and propagation terms have the same size, then  $T \simeq 1/\sigma^3$ . Regarding the nonlinear term, if it has the form  $\frac{\partial}{\partial x}(F \frac{\partial F}{\partial x})$  its order of magnitude is  $\sigma^2 F^2$ . The formation of the soliton requires that the nonlinear effect balances the dispersion. Hence it must have the same order of magnitude and  $F^2 \sigma^2 = F \sigma^4$ . Hence  $F \simeq \sigma^2$ . We can then conclude that  $F \ll L \ll D$  and the above equation describes the propagation of a wave with small amplitude ( $F$ ) and large wave length ( $L$ ) which travels large distances ( $D$ ). In the case of KP we have terms that describe the transverse evolution of the wave. We can estimate their sizes only if we make assumptions about the transverse length scales. In most cases the resulting flow is one-dimensional along the  $x$  direction with some “leakage” to the transverse directions.

## B. Numerical analysis

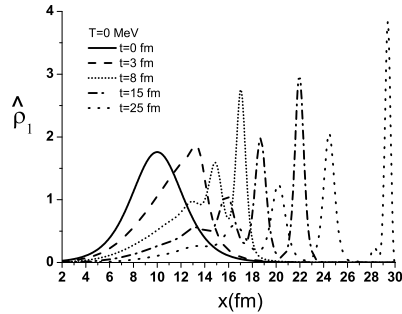
In what follows we apply the numerical tools developed in the appendix to several cases.

### 1. Nuclear soliton

We start our numerical analysis showing in Fig. 2 the solution of the linear KdV equation at  $T = 0$ , ( $\mathcal{G} = 0$  and  $\mathcal{X} = x$ ) (204) with  $d = \frac{1}{2}$ . In Fig. 2(a), we use the analytical solution (220) as initial condition. As expected this pulse propagates without dissipation nor dispersion: it is a soliton wave. This is the situation illustrated in Fig. 1. Any change in the initial condition has noticeable consequences as it can be seen in Fig. 2(b), where we follow the evolution of the numerical solution of (204) for an initial pulse given by (220) multiplied by a factor 20. As it can be seen, the amplitude grows, the width decreases and secondary bumps appear propagating behind the first.



(a)



(b)

FIG. 2: a) The evolution of the analytic solution of the KdV equation. b) The evolution of the analytic solution multiplied by a factor 20.

In Fig. 3, we show the equivalent plot for the spherical case:  $\mathcal{G} = 1$  and  $\mathcal{X} = r$ . In contrast to the linear case there is a strong damping of the pulse. The dependence on the initial conditions is also strong.

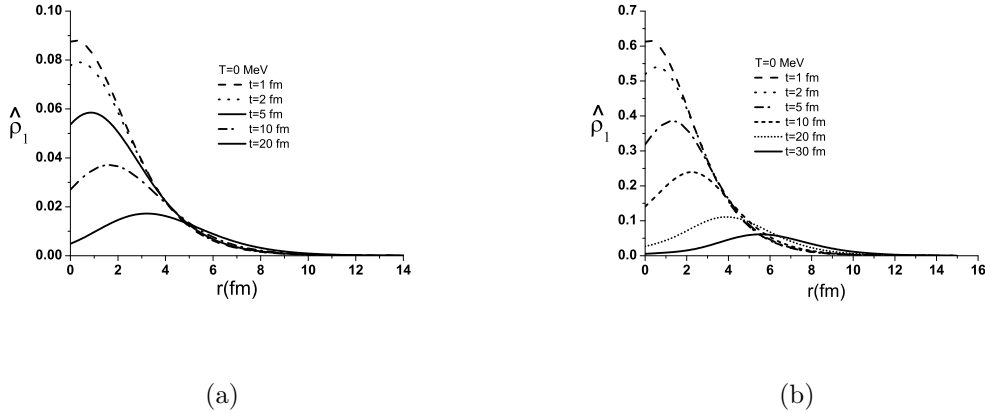


FIG. 3: Analogous to Fig. 2 for spherical coordinates.

In Fig. 4 we show for the linear case and for the “optimal” initial condition (220) the evolution of the pulse with time for different temperatures. We can see that, increasing the temperature the pulses move faster and go farther. The same feature can be observed in the spherical case, as shown in Fig. 5 .

Setting  $d = -\frac{1}{2}$  in the wave equations (204) and (207) we eliminate the third order derivative terms. The corresponding wave equations are breaking wave equations. Out of smooth initial perturbations, given by (220), these equations create shock waves. We can see this process in one dimensional Cartesian coordinates in Fig. 6 . We observe a steepening of the profile until the formation of the shock, followed by the dispersion of the wave. We see that the higher is the initial amplitude, the sooner the wave breaking and dispersion occurs.

In Fig. 7 we fix one initial profile and study its time evolution for two different tempe-

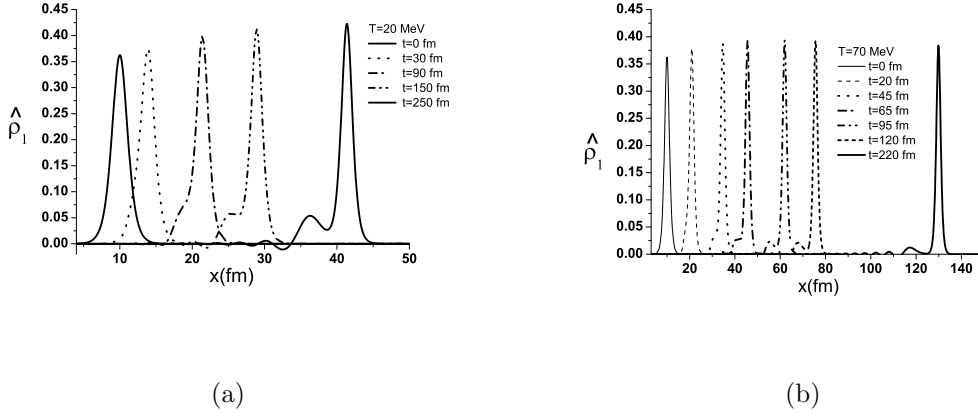


FIG. 4: The panels show calculations with temperatures a)  $T = 20$  and b)  $T = 70$  MeV .

atures. 7(a) and 7(b) show the development of a shock wave at  $T = 20$  MeV and 70 MeV respectively. As it can be seen, with increasing temperatures the pulse moves faster and the shock formation and the subsequent dispersive breaking occurs later. For the radial case a similar behavior is observed.

## 2. Breaking wave equation in QGP

The KdV equation can be written as:

$$\frac{\partial u}{\partial t} + c_s \frac{\partial u}{\partial t} + \beta \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) + \mu \frac{\partial^3 u}{\partial x^3} = 0 \quad (240)$$

which has the following analytical soliton solution:

$$f(x, t) = \frac{3(u - c_s)}{\beta} \operatorname{sech}^2 \left[ \sqrt{\frac{(u - c_s)}{4\mu}} (x - ut) \right] \quad (241)$$

In the numerical study of (208) and (209) we use the following soliton-like profile:

$$\hat{\rho}_1(x, t_0) = A \operatorname{sech}^2 \left[ \frac{x}{B} \right] \quad (242)$$

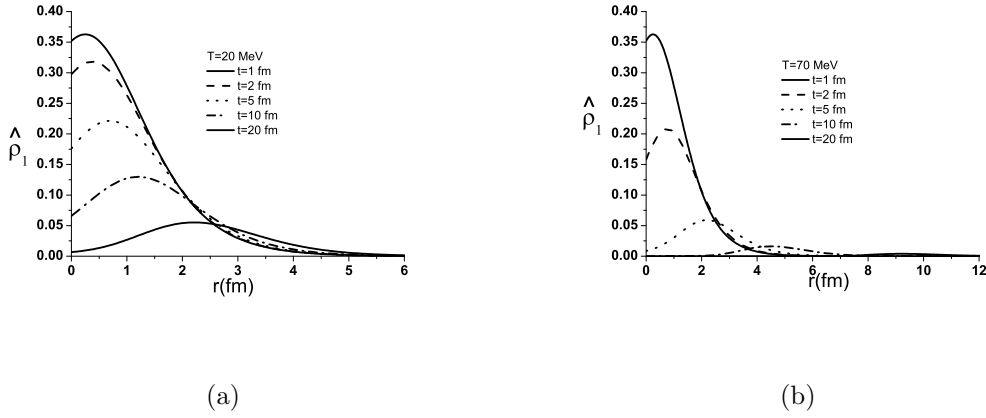


FIG. 5: Analogous to Fig. 4 for spherical coordinates

In this equation  $A$  and  $B$  represent the amplitude and width of the initial baryon density pulse, respectively.

We present in Fig. 8 the numerical solution of (208) for different times. In Fig. 8(a) we show for  $A = 0.075$  and  $B = 1$  fm and in 8(b) for  $A = 0.35$  and  $B = 1$  fm. It is possible to observe the evolution of the initial gaussian-like pulse with the formation of a “wall” on the right side. In 8(b) the “wall” formation and dispersion occurs much earlier than in 8(a) due higher initial amplitude.

In Fig. 9(a) we show the solution of (209) with the initial condition given by (242) with  $A = 0.01$ ,  $B = 1$  fm and  $T = 300$  MeV. Fig. 9(b) shows the same but with  $A = 0.1$  and  $B = 1$  fm. As in the zero temperature case, increasing the initial amplitude the breaking process and dispersion develops earlier. From Fig. 9(a) we can conclude that it is possible to find an approximate solitonic behavior even when the differential equation is not the KdV one.

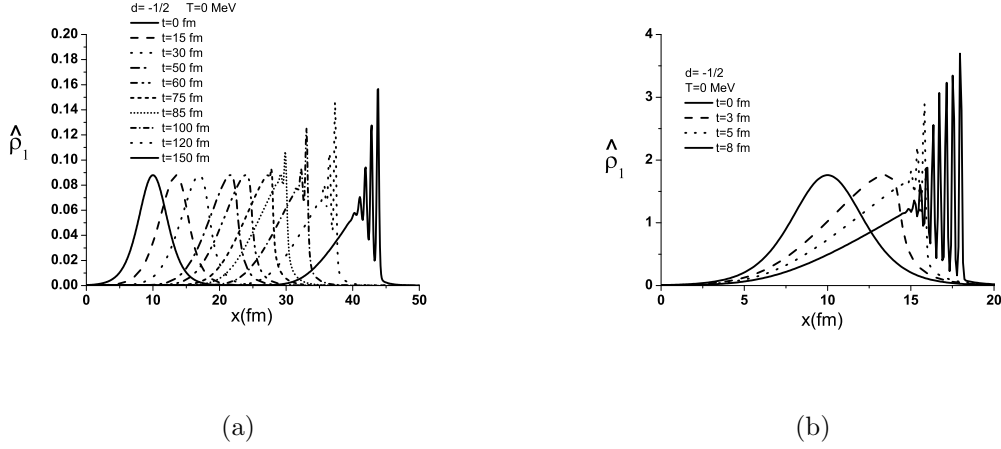


FIG. 6: a) Shock wave formation in one dimensional Cartesian coordinates. b) Initial profile multiplied by a factor 20.

### 3. KP equation in cold QGP

We now study the conditions in which the solution (223) must be real and therefore the constant  $h_1$  must be positive. Moreover, following Refs. [33, 34] we assume that  $a^2 + b^2 = 1$  and consider  $\hat{\rho}_1$  a normalized perturbation [42], within the region (in the  $u - a$  plane) Eq. (223) is well defined and we can have solitons. The parameters are chosen to be:  $\rho_0 = 1 \text{ fm}^{-3}$ ,  $g = 1.15$  and  $m_G = 460 \text{ MeV}$ , which imply  $c_s \simeq 0.64$  [42]. The stability analysis can be made more rigorous with the introduction of the Sagdeev potential [33, 34] by using  $\eta = \xi - ut = ar + bz - d \frac{c_s \varphi^2 t}{2} - ut$  to rewrite equation (212) as an energy balance equation. For our present purposes the requirements in [42] are sufficient.

The plot of the soliton evolution is presented in Fig. 10 and in Fig. 11. We show a plot of (223) with fixed  $\varphi = 0^\circ$ ,  $a = 0.6$ ,  $b = 0.8$ ,  $u = 0.73$  and  $z$  varying in the range  $0 \text{ fm} \leq z \leq 30 \text{ fm}$  which satisfies the soliton conditions. In Fig. 10(a) the pulse is observed



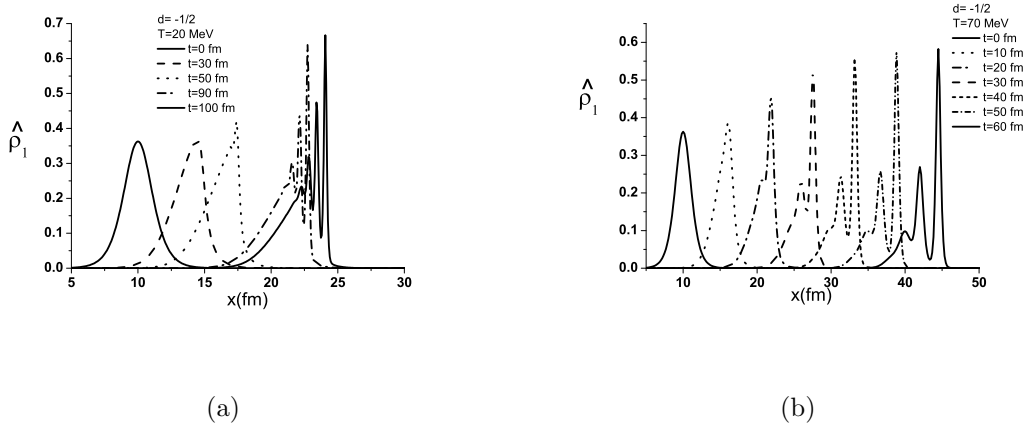


FIG. 7: Shock wave formation in one dimensional Cartesian coordinates for different temperatures: a)  $T = 20$  and b)  $T = 70$  MeV .

at  $t = 18$  fm whereas in Fig. 10(b) at  $t = 28$  fm. From the Fig. 10(a) we can see that the cylindrical pulse expands outwards in the radial direction. The regions with larger  $z$  expand with a delay with respect to the central ( $z = 0$ ) region.

Keeping  $z = 1$  fm fixed, we show the time evolution of (223) from  $t = 10$  fm (Fig. 11(a)) to  $t = 22$  fm (Fig. 11(b)). The azimuthal angle varies in the range  $20^\circ \leq \varphi \leq 150^\circ$ . From the parenthesis in (223) we can see that the expansion velocity grows with the angle. This asymmetry can be clearly seen in the figure, where the large angle “backward” region moves faster the small angle “forward” region. The breaking of  $z$  invariance and azimuthal symmetry is entangled with the soliton stability [42] and with the physical properties of the system (contained in the parameters  $h_1$ ,  $h_2$  and  $c_s$ ).

We perform the study of the existence condition for the solution (227), which must be real and therefore the constant  $U - w$  must be positive. The parameters are the same:  $\rho_0 = 1 \text{ fm}^{-3}$ ,  $g = 1.15$  and  $m_G = 460$  MeV, which imply  $c_s \simeq 0.64$  [42]. We also set  $\mathcal{C} = 0.5$

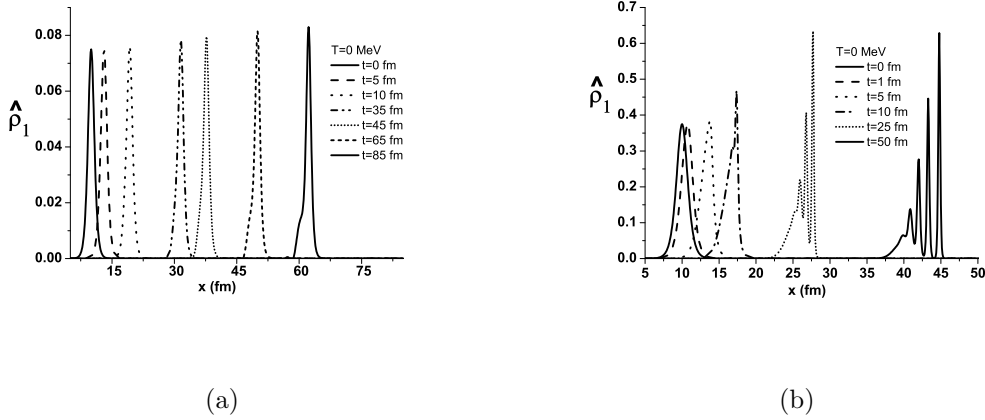
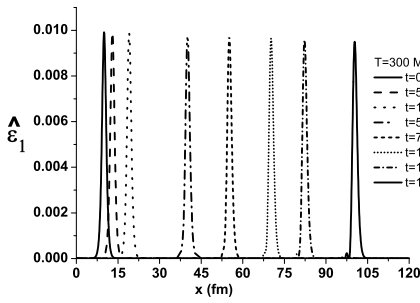


FIG. 8: Time evolution of a baryon density pulse for cold QGP.

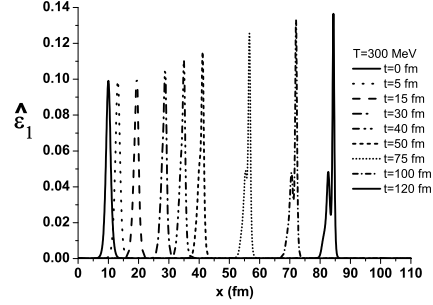
and extend the condition in Refs. [33, 34] to  $\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2 = 1$ . As mentioned before,  $U > w$  and again,  $\hat{\rho}_1$  is a normalized perturbation [42] and within the region (in the  $U - \mathcal{A}$  plane), (227) is well defined and we can have solitons [42]. The stability analysis can be performed more rigorously with the introduction of the Sagdeev potential [33, 34] by using  $\mathcal{A}x + \mathcal{B}y + \mathcal{C}z - Ut$ , to rewrite equation (213) as an energy balance equation. A simple example of soliton evolution is presented in Fig. 12. The plot of (227) with fixed  $z = 1$  fm,  $\mathcal{A} = 0.6$ ,  $\mathcal{B} \cong 0.62$ ,  $U = 0.66$  and  $y$  varying in the range  $0 \text{ fm} \leq y \leq 50 \text{ fm}$ . The pulse is observed at two times:  $t = 30$  fm (Fig. 12(a)) and  $t = 120$  fm (Fig. 12(b)). From the figure we can see that the cartesian pulse expands outwards in the  $x$  direction keeping its shape and form as expected.

Analogously to the nuclear soliton, the particular case of KP (213): the KdV (214), has the exact soliton solution (229). In Fig. 13 we choose  $u = 0.8$ ,  $\rho_0 = 2 \text{ fm}^{-3}$  and  $c_s^2 = 0.5$ .

In Fig. 13(a), the numerical solution of (214) for (229) as initial condition is studied for different times. As expected, the evolution of the initial gaussian-like pulse as a well



(a)



(b)

FIG. 9: Time evolution of an energy density pulse at  $T = 300 \text{ MeV}$ .

defined soliton, keeping its shape and form. In Fig. 13(b), we show again the numerical solution of (214) for (229) multiplied by a factor 10. Now the initial soliton possesses amplitude 6.0 and starts to develop secondary peaks, which are called “radiation” in the literature. Further time evolution would increase the strength of these peaks until the complete loss of localization.

For the other particular case (215) of the KdV (214), which we show in Fig. 14, where again we use (229) with the same parameters listed above. Fig. 14(a), we have the breaking following by its dispersion of the initial pulse. And in Fig. 14(b), we use (229) multiplied by a factor 10, which we observe the anticipation of the breaking following by its dispersion of the initial pulse much earlier in comparison to Fig. 14(a).

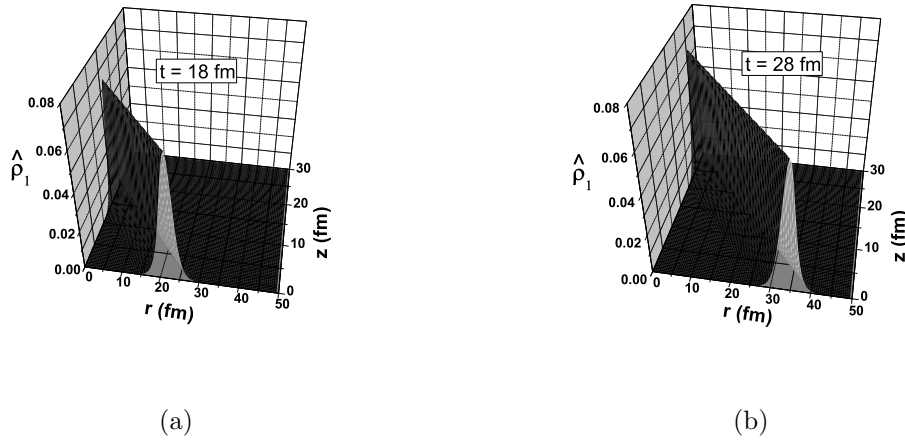


FIG. 10: Cylindrical soliton time evolution in  $r - z$  plane.

#### 4. Burgers equation in hot QGP

We apply the perturbation concept to study the radial expansion of cylindrical flux tubes in a hot QGP. These tubes are treated as perturbations in the energy density of the system which is formed in heavy ion collisions at RHIC and LHC as explained in [43]. During the expansion there is a “competition” between the background and the tube. In Fig. 15(a) the QGP background expands faster and the tube is “pushed” outwards generating an anisotropic energy (and final particle) distribution. In Fig. 15(b) the opposite occurs: the tube expands faster than the background, generating a different type of anisotropy in the final state. In principle two and three-particle correlation measurements could distinguish between the two cases. Viscosity may change this picture. As shown in [43], a strong viscosity could rapidly damp the tube and reduce any anisotropy. In Fig. 15 we have two extreme situations and something in-between may occur. With our formalism we can study quantitatively the evolution of the tube.

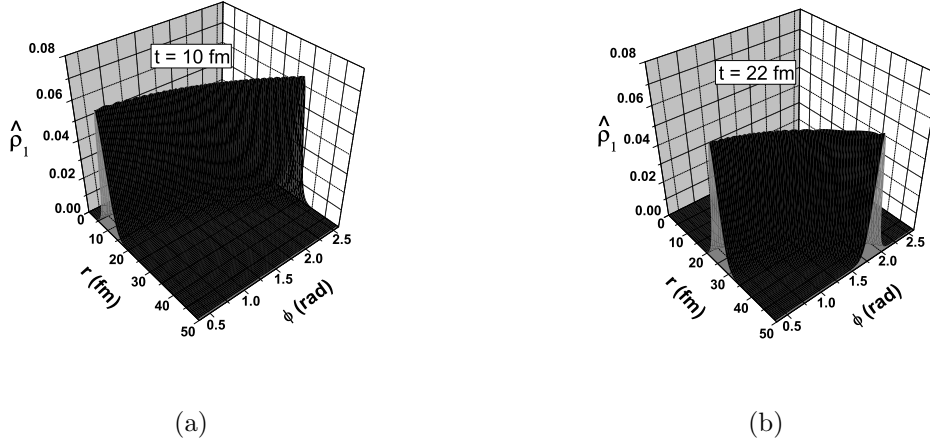


FIG. 11: Cylindrical soliton time evolution in  $r - \varphi$  plane.

We perform the numerical analysis for equations (218) and (219) with the initial condition given by a gaussian pulse in  $\hat{\varepsilon}_1$ :

$$\hat{\varepsilon}_1(r) = A e^{-r^2/r_0^2} \quad (243)$$

where the amplitude  $A$  and the approximate width  $r_0$  are parameters which depend on the dynamics of flux tube formation. We shall refer to  $r_0$  as the initial “radius” of the tube. The tubes are perturbations, so we expect  $A < 1$ . According to [43] and references therein, the transverse size of the tubes is of the order of 1 fm and in our calculations  $r_0 = 0.8 \text{ fm}$ . We consider hot QGP at temperatures  $T_0 = 150 \text{ MeV}$  and  $T_0 = 500 \text{ MeV}$  treated as an ideal fluid ( $\eta/s = \zeta/s = 0$ ) described by (219) and as a viscous fluid ( $\eta/s = 0.16$  and  $\zeta/s = 0$ ) described by (218) [43].

In Fig. 16 we show numerical solutions of (218) for a viscous fluid using (243) and we can observe the increasing temperature favors the tubular structure survival.

In Fig. 17 we perform the same study for the ideal fluid (219) with (243) and we show that breaking with dispersion occurs.

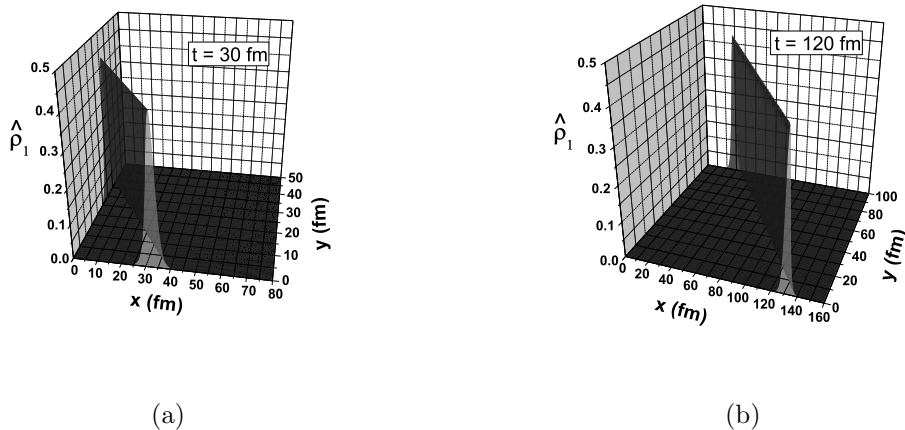


FIG. 12: Cartesian soliton time evolution in  $x - y$  plane.

By comparison between Fig. 16 and Fig. 17 we conclude that viscosity dissipates the breaking followed by dispersion of the pulse. The tube expands radially with a supersonic velocity and in less than  $4 \text{ fm}/c$  it becomes a “ring”, with a hole in the middle. Moreover, by this time the amplitude is already reduced by a factor two and the tube (or ring) loses the strength to “push away” the surrounding matter [43].

### 5. KP-Burgers in hot QGP

As an example of time evolution for the analytical solution of the cKP-B equation (217) we plot the time evolution of (232) with  $\varphi = 0 \text{ rad}$ ,  $D = 1$ ,  $A = B = 0.5$ ,  $T_0 = 300 \text{ MeV}$  and the viscous fluid with  $\eta/s = 0.16$  and  $\zeta/s = 0$ . The Fig. 18 shows the time evolution of the analytical traveling wave.

As it can be seen, (232) obtained depends directly of the dissipative (viscous) coefficient term of (217), i.e., it is a dissipation dominated solution. The Fig. 18 shows a shock wave

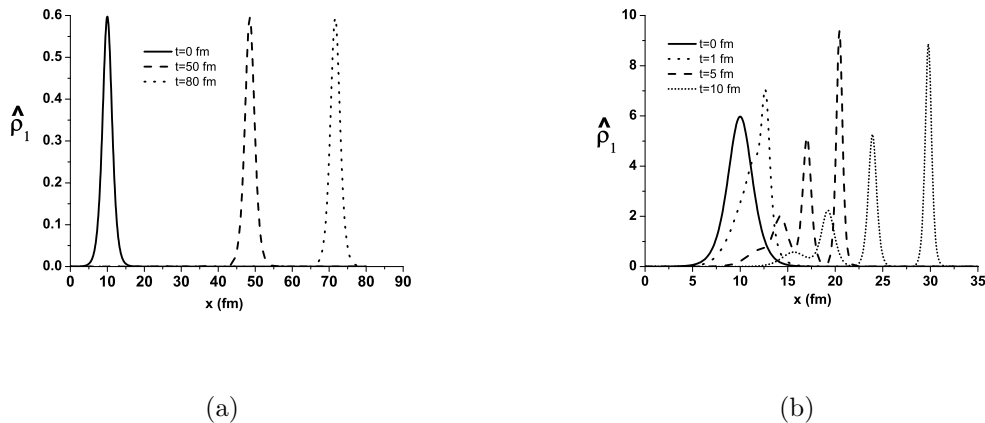


FIG. 13: a) Soliton propagation in one dimensional Cartesian coordinate and b) Initial profile multiplied by a factor 10 providing peaks plus “radiation”.

propagation in a time interval of 50  $fm$ .

### VIII. CONCLUSION

The discovery of the quark gluon plasma in the high energy heavy ion colliders brought relativistic hydrodynamics to the main stage of hadron physics. Encouraged by the vigorous experimental program at CERN theoreticians of hydrodynamics embarked in an ambitious project: the calculation of observables quantities with relativistic viscous hydrodynamics. The measurement of two and three particle correlations may be useful to study the propagation of waves in the QGP. Among the sources of waves we have fast partons crossing the medium and also flux tubes formed in the initial stage of heavy ion collisions. In this work we have emphasized that these waves are most likely nonlinear and should be studied with the appropriate formalism, which, in our opinion, is the Reductive Perturbation Method. After

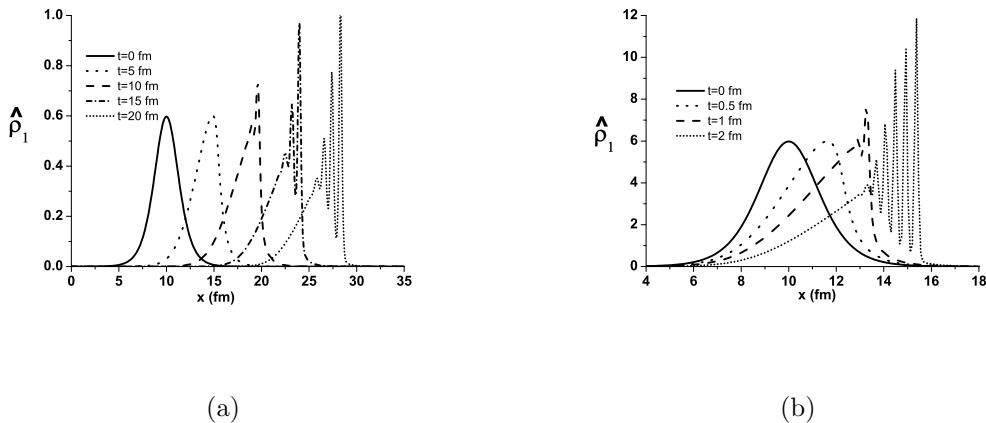


FIG. 14: a) Breaking wave and b) breaking wave anticipation.

making a survey of relativistic hydrodynamics we have presented the RPM in a simple and pedagogical way. The equation of state of the two most relevant strongly interacting fluids, i.e., nuclear matter and the quark gluon plasma, was discussed. In both cases we have made an effort to give a pedagogical introduction for non experts. In both cases we have shown how to obtain a KdV soliton. The main responsible for the appearance of these solitons are higher order derivative terms in the vector fields appearing in the Lagrangian of the system. In QHD it is enough to relax the strict mean field approximation, in which all gradients vanish, and allow for spatial inhomogeneities in the vector field. A slightly more careful treatment of the vector field equation of motion yields the desired term, derived from the Laplacian  $\nabla^2 V_0$ . In QCD, the situation is more complicated because the gluon is massless. This makes impossible a simple estimate of the corresponding quantity  $\nabla^2 A_0^a$ . However a careful treatment of the non-vanishing vacuum condensates leads to a dynamically generated gluon mass,  $m_G$ , which introduces a mass and a size scale and renders possible the estimate of the desired Laplacian and the existence of solitons in the QGP.



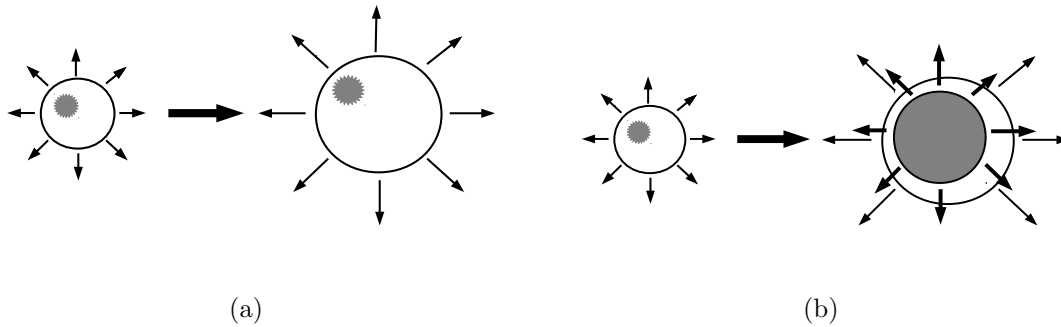


FIG. 15: The large circle represents the front view of a cylindrical portion of QGP fluid. The small and dark circle represents a flux tube, i.e., a cylindrical perturbation with a higher energy density than the background. a) The tube expands slowly and the background faster. b) The tube expands much faster than the background.

Combining the equations of hydrodynamics with the equation of state and applying the RPM we have derived several differential equations for the perturbations in energy and baryon density. These equations connect properties of the waves, such as width and speed, with the microscopic dynamical quantities of the fluids, such as particle masses and couplings. Several of them have analytical solutions, which were presented and discussed. Some others must be solved numerically. As expected for nonlinear equations, the results depend very strongly on the initial conditions. An interesting finding is that, even when we do not have a KdV equation, in many cases the breaking wave equation has very long living localized solutions, which resemble to solitons. In some other cases, the initial pulses loose their localization and/or start to present rapid oscillations. All these features may manifest themselves directly or indirectly in the experimental data. The analysis made here

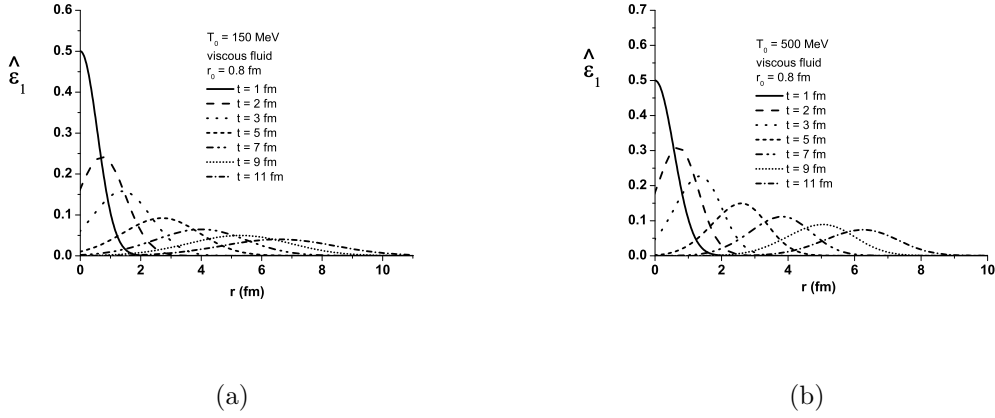


FIG. 16: Solutions for several times: a)  $T_0 = 150 \text{ MeV}$  and b)  $T_0 = 500 \text{ MeV}$ .

is still qualitative and a closer contact with phenomenology is still to be made. For now, the obtained results suggest that viscosity strongly affects the propagation of perturbations in the quark gluon plasma. In order to confirm this statement the next step is to apply the RPM to the Müller-Israel-Stewart theory.

We hope to have convinced the reader that the study of nonlinear waves in hadron physics is an interesting and fast moving field. This study will help to interpret and understand the data from the LHC.

## IX. APPENDIX: METHOD OF FINITE DIFFERENCES

The most general form of a one-dimensional nonlinear wave equation with a second order dissipative term and a third order dispersive terms is given by:

$$\frac{\partial u}{\partial t} + \beta \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) + \alpha \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^3 u}{\partial x^3} = 0 \quad (244)$$

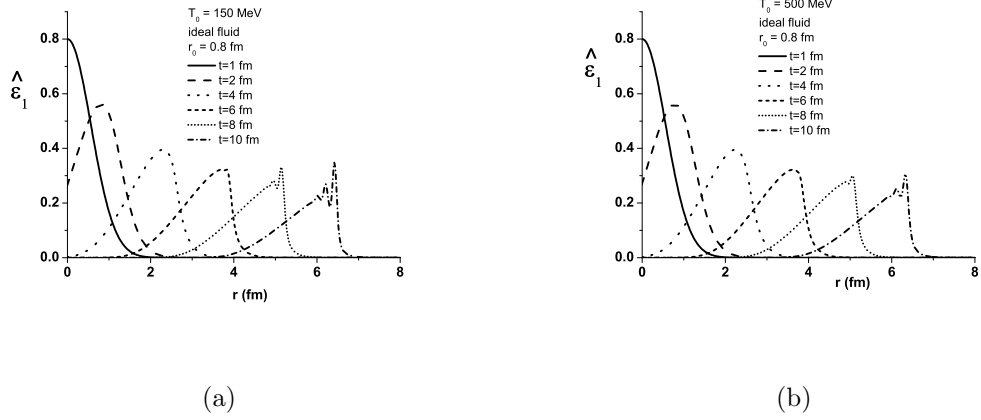


FIG. 17: Analogous to Fig. 16 but for the ideal fluid. a)  $T_0 = 150 \text{ MeV}$  and b)  $T_0 = 500 \text{ MeV}$ .

For  $\alpha = 0$ ,  $\beta$  and  $\mu \neq 0$ , it is the usual third order nonlinear Kortweg-de Vries equation (KdV). For  $\mu = 0$ ,  $\beta$  and  $\alpha \neq 0$ , it represents the Burgers equations. Finally, when only  $\beta \neq 0$ , we have the breaking wave equation. In order to solve this equation numerically, we divide the integration region ( $0 \leq x \leq (n+1)h$  and  $0 \leq t \leq m\Delta t$ ) with a space step  $h$  and a time step  $\Delta t$ . Then, the wave function  $u(x, t)$ , solution of equation (244), assume the discrete values  $u_{i,0}, u_{i,1}, \dots, u_{i,j-2}, u_{i,j-1}, u_{i,j}, u_{i,j+1}, u_{i,j+2}, \dots, u_{i,n}, u_{i,n+1}$  in a given time  $t_i$ . The expansion of this solution in Taylor's series leads to:

$$u_{i,j+1} = u_{i,j} + \left( \frac{\partial u}{\partial x} \right)_{i,j} h + \frac{1}{2!} \left( \frac{\partial^2 u}{\partial x^2} \right)_{i,j} h^2 + \frac{1}{3!} \left( \frac{\partial^3 u}{\partial x^3} \right)_{i,j} h^3 + O(h^4) \quad (245)$$

or, alternatively, to:

$$u_{i,j-1} = u_{i,j} - \left( \frac{\partial u}{\partial x} \right)_{i,j} h + \frac{1}{2!} \left( \frac{\partial^2 u}{\partial x^2} \right)_{i,j} h^2 - \frac{1}{3!} \left( \frac{\partial^3 u}{\partial x^3} \right)_{i,j} h^3 + O(h^4) \quad (246)$$

We combine these expressions conveniently to obtain finite difference expressions for first,

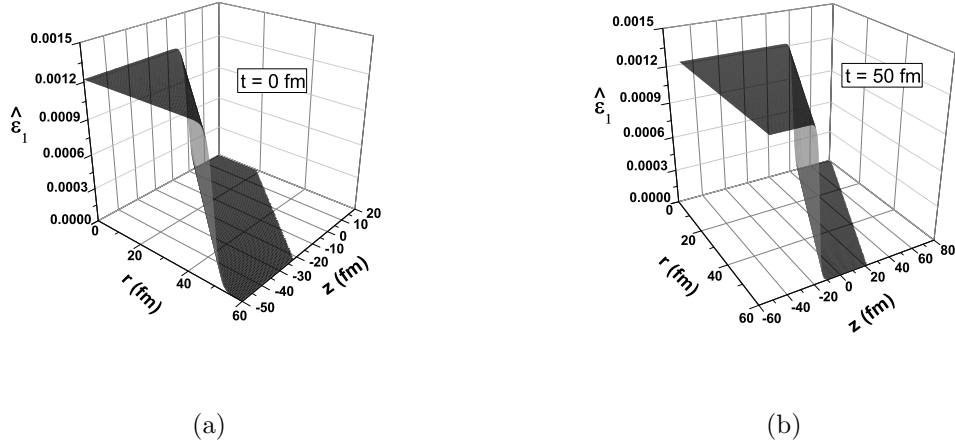


FIG. 18: Shock wave evolution: a)  $t = 0 \text{ fm}$  and b)  $t = 50 \text{ fm}$ .

second and third order centered partial space derivatives [61]:

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2h} = O_x U_i, \quad j = 2, \dots, n-1 \quad (247)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = O_{xx} U_i \quad j = 2, \dots, n-1 \quad (248)$$

and

$$\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \approx \frac{-u_{i,j+2} + 2u_{i,j+1} - 2u_{i,j-1} + u_{i,j-2}}{2h^3} = O_{xxx} U_i \quad j = 2, \dots, n-1 \quad (249)$$

in which we define

$$U_i = \begin{pmatrix} u_{i,2} \\ \vdots \\ u_{i,j+1} \\ u_{i,j} \\ \vdots \\ u_{i,n-1} \end{pmatrix} \quad (250)$$

( $u_{i,0} = u_{i,1} = u_{i,n} = u_{i,n+1} = 0$  are the boundary values on  $x$  axe) and the operators:

$$O_x = \begin{pmatrix} 0 & 1/2h & 0 & \cdots & 0 \\ -1/2h & 0 & 1/2h & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1/2h & 0 & 1/2h \\ 0 & \cdots & 0 & -1/2h & 0 \end{pmatrix} \quad (251)$$

$$O_{xx} = \begin{pmatrix} -2/h^2 & 1/h^2 & 0 & \cdots & 0 \\ 1/h^2 & -2/h^2 & 1/h^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1/h^2 & -2/h^2 & 1/h^2 \\ 0 & \cdots & 0 & 1/h^2 & -2/h^2 \end{pmatrix} \quad (252)$$

and

$$O_{xxx} = \begin{pmatrix} 0 & -1/h^3 & 1/2h^3 & 0 & 0 & \cdots & 0 \\ 1/h^3 & 0 & -1/h^3 & 1/2h^3 & 0 & \cdots & 0 \\ -1/2h^3 & 1/h^3 & 0 & -1/h^3 & 1/2h^3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1/2h^3 & 1/h^3 & 0 & -1/h^3 & 1/2h^3 \\ 0 & \cdots & 0 & -1/2h^3 & 1/h^3 & 0 & -1/h^3 \\ 0 & \cdots & 0 & 0 & -1/2h^3 & 1/h^3 & 0 \end{pmatrix} \quad (253)$$

Analogously, we can expand the solution of equation (244) around a time  $t_i$ :

$$U_{i+1} = U_i + \left( \frac{\partial u}{\partial t} \right)_{i,j} \Delta t + \frac{1}{2} \left( \frac{\partial^2 u}{\partial t^2} \right)_{i,j} (\Delta t)^2 + O(h^3) \quad (254)$$

to obtain

$$\left( \frac{\partial u}{\partial t} \right)_{i,j} \approx \frac{U_{i+1} - U_i}{\Delta t} \quad (255)$$

When we apply (255) and the operators (247), (248) and (249) in equation (244), it can be represented equally in times  $t_i$  or  $t_{i+1}$ , i.e., the following:

$$\frac{U_{i+1} - U_i}{\Delta t} = -\beta O_x[(1/2)U_i^2] - \alpha O_{xx}U_i - \mu O_{xxx}U_i \quad (256)$$

or

$$\frac{U_{i+1} - U_i}{\Delta t} = -\beta O_x[(1/2)U_{i+1}^2] - \alpha O_{xx}U_{i+1} - \mu O_{xxx}U_{i+1} \quad (257)$$

As long as the error one makes in both situations is the same, the Crank-Nicolson scheme [62] prescribes to take the mean value of these two possibilities, therefore

$$U_{i+1} = U_i - \frac{1}{2}\Delta t \beta O_x [(1/2)U_i^2 + (1/2)U_{i+1}^2] - \frac{1}{2}\Delta t \alpha O_{xx}(U_{i+1} + U_i) - \frac{1}{2}\Delta t \mu O_{xxx}(U_{i+1} + U_i) \quad (258)$$

Given a initial condition  $U_0$ ,  $U_{i+1}$  is the solution of (258) at any posterior time. However, this is a set of nonlinear algebraic equations. This problem becomes much more simple if we linearize it. We expand the nonlinear term in a Taylor's series, around the  $i - th$  time  $t_i$  [63]:

$$\frac{1}{2}U_{i+1}^2 = \frac{1}{2}U_i^2 + \Delta t \left( \frac{\partial(1/2)u^2}{\partial u} \right)_i \left( \frac{\partial u}{\partial t} \right)_i + O(\Delta t^2) \quad (259)$$

which leads to

$$\frac{1}{2}U_{i+1}^2 = \frac{1}{2}U_i^2 + U_i(U_{i+1} - U_i) \quad (260)$$

where we have used (255).

Replacing this result in (258) we get:

$$U_{i+1} = U_i - \frac{1}{2}\Delta t \beta O_x(U_i U_{i+1}) - \frac{1}{2}\Delta t \alpha O_{xx}(U_{i+1} + U_i) - \frac{1}{2}\Delta t \mu O_{xxx}(U_{i+1} + U_i) \quad (261)$$

Using the matrix format of the operators  $O_x$  (251),  $O_{xx}$  (252) and  $O_{xxx}$  (253), equation (261) becomes a quin-diagonal algorithm:

$$\begin{pmatrix} 1-2s & -q_{i,j}+s & -p & 0 & 0 & \cdots & 0 \\ q_{i,j}+s & 1-2s & -q_{i,j}+s & -p & 0 & \cdots & 0 \\ p & q_{i,j}+s & 1-2s & -q_{i,j}+s & -p & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & p & q_{i,j}+s & 1-2s & -q_{i,j}+s & -p \\ 0 & \cdots & 0 & p & q_{i,j}+s & 1-2s & -q_{i,j}+s \\ 0 & \cdots & 0 & 0 & p & q_{i,j}+s & 1-2s \end{pmatrix} \begin{pmatrix} u_{i+1,2} \\ \vdots \\ u_{i+1,j-1} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ \vdots \\ u_{i+1,n-1} \end{pmatrix} = \begin{pmatrix} r_{i,2} \\ \vdots \\ r_{i,j-1} \\ r_{i,j} \\ r_{i,j+1} \\ \vdots \\ r_{i,n-1} \end{pmatrix} \quad (262)$$

in which

$$p = -\frac{1}{4}\mu \frac{\Delta t}{h^3} \quad (263)$$

$$q_{i,j} = -\frac{1}{4}\beta \frac{\Delta t}{h} u_{i,j-1} - 2p \quad (264)$$

$$s = \frac{1}{2}\alpha \frac{\Delta t}{h^2} \quad (265)$$

and

$$r_{i,j} = (1 + 2s)u_{i,j} + pu_{i,j+2} - (2p + s)u_{i,j+1} + (2p - s)u_{i,j-1} - pu_{i,j-2} \quad (266)$$

Therefore, from a given initial condition:

$$U_0 = \begin{pmatrix} u_{0,2} \\ \vdots \\ u_{0,j-1} \\ u_{0,j} \\ u_{0,j+1} \\ \vdots \\ u_{0,n-1} \end{pmatrix} \quad (267)$$

the set of linear algebraic equations (262) can be iteratively solved, to obtain the solution  $u(x, t)$  of equation (244) in any posterior time  $t_{i+1}$  is given by:

$$U_{i+1} = \begin{pmatrix} u_{i+1,2} \\ \vdots \\ u_{i+1,j-1} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ \vdots \\ u_{i+1,n-1} \end{pmatrix} \quad (268)$$

We turn now to the two dimensional extension of the KdV equation (237), the so called Kadomtsev-Petviashvilli (KP) equation (236):

$$\frac{\partial}{\partial x} \left\{ \frac{\partial u}{\partial t} + \frac{\alpha_1}{2} \frac{\partial u^2}{\partial x} + \alpha_2 \frac{\partial^3 u}{\partial x^3} \right\} + \alpha_3 \frac{\partial^2 u}{\partial y^2} = 0 \quad (269)$$

Repeating all the preceding procedure for this equation, we obtain:

$$O_x U_{i+1} = O_x U_i - \frac{1}{2} \Delta t \alpha_1 O_{xx} (U_i U_{i+1}) - \frac{1}{2} \Delta t \alpha_2 O_{xxx} (U_{i+1} + U_i) - \frac{1}{2} \Delta t \alpha_3 O_{yy} (U_{i+1} + U_i) \quad (270)$$

in which we define,

$$U_i = \begin{pmatrix} u_{i,2,k} \\ \vdots \\ u_{i,j+1,k} \\ u_{i,j,k} \\ \vdots \\ u_{i,n-1,k} \end{pmatrix} \quad k = 1, 2, \dots, l \quad (271)$$

( $u_{i,j,0} = u_{i,j,l+1} = 0$  are the boundary values in the  $y$  direction) and the operators [61]:

$$O_{xxxx} = \begin{pmatrix} 6/h^4 & -4/h^4 & 1/h^4 & 0 & 0 & \dots & 0 \\ -4/h^4 & 6/h^4 & -4/h^4 & 1/h^4 & 0 & \dots & 0 \\ 1/h^4 & -4/h^4 & 6/h^4 & -4/h^4 & 1/h^4 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1/h^4 & -4/h^4 & 6/h^4 & -4/h^4 & 1/h^4 \\ 0 & \dots & 0 & 1/h^4 & -4/h^4 & 6/h^4 & -4/h^4 \\ 0 & \dots & 0 & 0 & 1/h^4 & -4/h^4 & 6/h^4 \end{pmatrix} \quad (272)$$

is the finite differences fourth order centered partial space derivative operator, and

$$O_{yy} = \begin{pmatrix} -2/h_y^2 & 1/h_y^2 & 0 & \dots & 0 \\ 1/h_y^2 & -2/h_y^2 & 1/h_y^2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1/h_y^2 & -2/h_y^2 & 1/h_y^2 \\ 0 & \dots & 0 & 1/h_y^2 & -2/h_y^2 \end{pmatrix} \quad (273)$$

in which  $h_y$  is the space step in the  $y$  direction.

Replacing the matrix representation of the operators in equation (270) we get:

$$\begin{pmatrix} c_{i,j,k} & d_{i,j,k} & a & 0 & 0 & \dots & 0 \\ b_{i,j,k} & c_{i,j,k} & d_{i,j,k} & a & 0 & \dots & 0 \\ a & b_{i,j,k} & c_{i,j,k} & d_{i,j,k} & a & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a & b_{i,j,k} & c_{i,j,k} & d_{i,j,k} & a \\ 0 & \dots & 0 & a & b_{i,j,k} & c_{i,j,k} & d_{i,j,k} \\ 0 & \dots & 0 & 0 & a & b_{i,j,k} & c_{i,j,k} \end{pmatrix} \begin{pmatrix} u_{i+1,2,k} \\ \vdots \\ u_{i+1,j-1,k} \\ u_{i+1,j,k} \\ u_{i+1,j+1,k} \\ \vdots \\ u_{i+1,n-1,k} \end{pmatrix} = \begin{pmatrix} e_{i,2,k} \\ \vdots \\ e_{i,j-1,k} \\ e_{i,j,k} \\ e_{i,j+1,k} \\ \vdots \\ e_{i,n-1,k} \end{pmatrix} \quad k = 1, 2, \dots, l \quad (274)$$



in which

$$a = \alpha_2 \frac{\Delta t}{h^3} \quad (275)$$

$$b_{i,j,k} = 1 - 4a + \frac{\alpha_1 \Delta t}{h} u_{i,j+1,k} \quad (276)$$

$$c_{i,j,k} = 6a - \frac{2\alpha_3 h \Delta t}{h_y^2} - \frac{2\alpha_1 \Delta t}{h} u_{i,j,k} \quad (277)$$

$$d_{i,j,k} = -1 - 4a + \frac{\alpha_1 \Delta t}{h} u_{i,j-1,k} \quad (278)$$

$$e_{i,j,k} = -a u_{i,j+2,k} + (1 + 4a) u_{i,j+1,k} + \left( \frac{2\alpha_3 h \Delta t}{h_y^2} - 6a \right) u_{i,j,k} + (-1 + 4a) u_{i,j-1,k} - a u_{i,j-2,k} \\ - \frac{2\alpha_3 h \Delta t}{h_y^2} (u_{i+1,j,k+1} + u_{i+1,j,k-1} + u_{i,j,k+1} + u_{i,j,k-1})$$

The algorithm (274) represents  $l$  sets of linear algebraic equations. As long as each one of these sets are self-consistent, they must be solved iteratively, until the answer converges to the solution  $(U^{i+1})^f$  after  $f$  iterations. To stop the iterations, we can use the criterion [64]:

$$\frac{\|(U^{i+1})^f - (U^{i+1})^{f-1}\|}{\|(U^{i+1})^{f-1}\|} < \epsilon \quad (279)$$

In what concerns the stability of this numerical method, the conservation of some quantities such as:

$$P(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u dx dy \quad (280)$$

and

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^2 dx dy \quad (281)$$

are frequently used as a criterion to verify its reliability. In refs. [65] and [66], this criterion is used to show analytically that these numerical methods based on the Crank-Nicolson scheme are unconditionally stable.

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